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A characterization of the Schur property through the disk algebra ☆

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This paper is dedicated to our dear friend Richard Aron

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ABSTRACT

In this paper we give a new characterization of when a Banach space E has the Schur property in terms of the disk algebra. We prove that E has the Schur property if and only if $A(\mathbb{D}, E) = A(\mathbb{D}, E_w)$.

1. Introduction

The disk algebra, whether for a single, finitely many, or infinite variables is an area of intensive research (see e.g. [1–5,9,11–15]). In this paper we consider the natural vector-valued extension of the disk algebra $A(\mathbb{D})$.

Let X and E be complex Banach spaces. As usual, B_X and $\overline{B}_X$ will stand for the open (respectively closed) unit ball of X. By $H(B_X, E)$ we denote the space of all mappings $f : B_X \to E$ holomorphic (i.e. complex-Fréchet differentiable) on B_X. As in the scalar valued case, the vector-valued extension of the disk algebra has two natural and equivalent definitions. One, denoted by $A_u(B_X, E)$, is the Banach space of all uniformly continuous functions $f : B_X \to E$ that, moreover, are holomorphic on B_X, endowed with the supremum norm. The other natural definition is the following.

$$A_u(\overline{B}_X, E) := \{f : \overline{B}_X \to E : f \in H(B_X, E) \text{ and } f \text{ uniformly continuous on } \overline{B}_X\}.$$

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Clearly the mapping $R : A_u(\overline{B}_X, E) \to A_u(B_X, E)$ that associates to each element in $A_u(\overline{B}_X, E)$ its restriction to the open unit ball B_X is an isometric isomorphism, since uniformly continuous functions defined on the open unit ball B_X of a Banach space X and with values in another Banach space are bounded and admit a unique extension to the closed unit ball $\overline{B}_X$ which is also uniformly continuous. Thus, from now on, we write $A_u(B_X, E) = A_u(\overline{B}_X, E)$. For $\mathbb{C}$-valued functions we simply denote $A_u(\overline{B}_X, \mathbb{C}) = A_u(B_X)$.

With E_τ we denote E endowed with the topology τ which is either the weak topology $w(E, E^*)$ or, whenever E is a dual space, i.e. there exists a complex Banach space Y such that $E = Y^*$, the weak-star topology $w^*(Y^*, Y)$.

A very classical result by Dunford of 1938 [6, Theorem 76, p. 354] or [10, Theorem 3.10.1, p. 93 combined with Theorem 3.17.1, p. 112], states that $H(B_X, E_w) = H(B_X, E)$. This means that a mapping $f : B_X \to E$ is holomorphic if and only if $u \circ f : B_X \to \mathbb{C}$ is holomorphic for every $u : E \to \mathbb{C}$ continuous linear form (in short for every $u \in E^*$).

Moreover, if $E = Y^*$, then $H(B_X, E_{w*}) = H(B_X, E)$. Again a mapping $f : B_X \to Y^*$ is holomorphic if and only if $u \circ f : B_X \to \mathbb{C}$ is holomorphic for every $u \in Y$ where we consider Y as a subspace of $E^* = Y^{**}$.

The main goal of this paper is to discuss if analogues of Dunford's results are true in the context of vector-valued algebras of the disk (or more properly called, algebras of the ball).

For that reason, we are going to consider the following spaces.

$$A_u(\overline{B}_X, E_\tau) := \{f : \overline{B}_X \to E : f \in H(B_X, E) \text{ and } f \text{ is } \tau - \text{uniformly continuous on } \overline{B}_X\},$$

and

$$A_u(B_X, E_\tau) := \{f : B_X \to E : f \in H(B_X, E) \text{ and } f \text{ is } \tau - \text{uniformly continuous on } B_X\},$$

where τ denotes either the topology w or w^*. Observe that when considering the norm topology in the range space, we simply write E. All of these spaces are Banach spaces when endowed with the supremum norm topology.

We explore the connections between these algebras of the disk,

$$A_u(\overline{B}_X, E) = A_u(B_X, E), \qquad A_u(B_X, E_w)$$

and the space of mapping defined in the closed unit ball $A_u(\overline{B}_X, E_w)$. Since the mapping $R : A_u(\overline{B}_X, E_w) \to A_u(B_X, E_w)$ defined as $R(f)(x) = f(x)$ for every x in B_X is well defined, injective, and actually an isometry into, one can consider $A_u(\overline{B}_X, E_w)$ as a subset of $A_u(B_X, E_w)$, and we have the following chain of inclusions.

$$A_u(B_X, E) = A_u(\overline{B}_X, E) \subseteq A_u(\overline{B}_X, E_w) \subseteq A_u(B_X, E_w). \tag{1.1}$$

Contrary to the Dunford's first stated result for holomorphic mapping both inclusions can be strict. This claim is shown in Section 2, where in Theorem 2.3 a necessary and sufficient condition for the equality $A_u(\overline{B}_X, E_w) = A_u(B_X, E_w)$ is given. Moreover, our main result, Theorem 2.7, proves that given a complex Banach space X, the equality $A_u(B_X, E) = A_u(B_X, E_w)$ holds if and only if E has the Schur property. Therefore, we give a new characterization of that property. We recall that a Banach space E has the *Schur property* if every weakly convergent sequence is norm convergent (see [7, p. 253]). The classical Banach sequence space ℓ_1 has this property [7, Theorem 5.36].

In Section 3 we give two different sufficient conditions for the Banach space $A_u(B_X, E_w)$ to be a Banach algebra whenever the space E is a Banach algebra.

We refer to [7] for notation and background information on Banach spaces. We will use the following classical Banach sequence spaces. The space c_0 of all null sequences endowed with the supremum norm, the space ℓ_∞ of all bounded sequences also endowed with the supremum norm and the space ℓ_1 of all absolutely summable sequences $(x_n)_n$ endowed with the usual norm given by $\|(x_n)_n\| := \sum_{n=1}^{\infty} |x_n|$, $(x_n)_n \in \ell_1$.