J. Math. Anal. Appl. 445 (2017) 1321–1327

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Renormings concerning the lineability of the norm-attaining functionals

Francisco Javier García-Pacheco [a,*], Daniele Puglisi [b]

[a] Department of Mathematics, University of Cadiz, 11510 Puerto Real, Spain
[b] Department of Mathematics, University of Catania, 95135 Catania, Italy

ARTICLE INFO

Article history:
Received 16 November 2015
Available online 27 January 2016
Submitted by D. Ryabogin

Keywords:
Lineability
Norm-attaining functional
Renorming
Norming
Biorthogonal

ABSTRACT

We prove that if a Banach space admits a biorthogonal system whose dual part is norming, then the set of norm-attaining functionals is lineable. As a consequence, if a Banach space admits a biorthogonal system whose dual part is bounded and its weak-star closed absolutely convex hull is a generator system, then the Banach space can be equivalently renormed so that the set of norm-attaining functionals is lineable. Finally, we prove that every infinite dimensional separable Banach space whose dual unit ball is weak-star separable has a linearly independent, countable, weak-star dense subset in its dual unit ball. As a consequence, we show the existence of linearly independent norming sets which are not the dual part of a biorthogonal system.

1. Introduction

Filling subspaces of ℓ_∞ have been studied in [9] where it is shown that, whereas not every subspace of ℓ_∞ verifies that the set of its norm-attaining functionals is lineable, filling subspaces do.

Following the notion of a "big set" in the measure theory sense (the complementary of a measure zero set) and in the Baire theory sense (a comeager set), Gurariy coined in 1991 (see [11]) a new version of this notion in the linear sense: *lineability* and *spaceability*. However, this did not appear in the literature until the early 2000's in [3,12]. For the last decade there has been an intensive trend to search for large algebraic and linear structures of special objects. We would like to mention the nice survey paper [6] related to this topic and the very recent monograph [2]. Let us introduce what we are meaning: A subset M of a Banach space X is said to be *lineable* (*spaceable*) if $M \cup \{0\}$ contains an infinite dimensional (closed) vector subspace. By λ-lineable (λ-spaceable) we mean that $M \cup \{0\}$ contains a (closed) vector subspace of dimension λ.

Throughout this paper, we will deal with a special friend: $\mathsf{NA}(X)$, the set of norm-attaining functionals on a Banach space X. By a classical Bishop–Phelps's theorem it is known that $\mathsf{NA}(X)$ is always "topologically

* Corresponding author.
E-mail addresses: garcia.pacheco@uca.es (F.J. García-Pacheco), dpuglisi@dmi.unict.it (D. Puglisi).

http://dx.doi.org/10.1016/j.jmaa.2016.01.052
0022-247X/

generic", that is, dense in X^*, therefore it seems natural to raise the following question (originally posed by Godefroy in [10]).

Problem 1.1. (See Godefroy, [10].) Given an infinite dimensional Banach space X, is $\mathsf{NA}(X)$ always lineable?

Very recently, Rmoutil in [14] observed that the example of Read [13] of a Banach space with no proximinal subspaces of codimension 2 is also an example of a Banach space whose set of norm-attaining functionals does not contain subspaces of dimension 2. In [1] it has been shown that the above question has a positive answer for some classical Banach spaces like the $\mathcal{C}(K)$ and the $\mathsf{L}_1(\mu)$ spaces. Concerning Question 1.1 in terms of spaceability, the main effort has been done by Bandyopadhyay and Godefroy in [5], where it was shown that Asplund Banach spaces with the Dunford–Pettis property cannot be equivalently renormed to make the norm-attaining functionals spaceable. In particular, if K is an infinite Hausdorff scattered compact topological space, then $\mathsf{NA}(\mathcal{C}(K))$ is lineable but not spaceable.

As far as we know, the main result obtained until now concerning the isomorphic lineability of $\mathsf{NA}(X)$ was obtained in [8], where it is shown that every Banach space admitting an infinite dimensional separable quotient can be equivalently renormed so that the set of its norm-attaining functionals is lineable.

All the Banach spaces throughout this manuscript will be considered infinite dimensional.

2. Filling subspaces of $\ell_\infty(\Lambda)$

We refer the reader to [9, Definition 2.4] for the original definition of filing subspace of ℓ_∞. Here we will generalize it for $\ell_\infty(\Lambda)$. From now on and unless explicitly stated, Λ will stand for an infinite set.

Given a subset V of $\ell_\infty(\Lambda)$, we define the supporting set of V as

$$\mathrm{supp}(V) := \bigcup \{\mathrm{supp}(v) : v \in V\},$$

where as expected $\mathrm{supp}(v) := \{\lambda \in \Lambda : v(\lambda) \neq 0\}$. Observe that if $\mathrm{supp}(V)$ is finite and V is a subspace, then V is finite dimensional.

An infinite dimensional closed subspace V of $\ell_\infty(\Lambda)$ is said to be filling provided that for every infinite subset A of $\mathrm{supp}(V)$ there exists $x \in \mathsf{S}_V$ with $\mathrm{supp}(x) \subseteq A$ and x attains its sup norm.

It is not hard to see that every infinite dimensional closed subspace of $\ell_\infty(\Lambda)$ containing $c_{00}(\Lambda)$ is filling.

Also recall that for every $\lambda \in \Lambda$, the evaluation functional δ_λ on $\ell_\infty(\Lambda)$ is defined by $\delta_\lambda(x) = x(\lambda)$. It is not hard to see that $\delta_\lambda \in \mathsf{S}_{\ell_\infty(\Lambda)^*}$, and if $\lambda_1, \dots, \lambda_p \in \Lambda$ are all different, then

$$\|\alpha_1 \delta_{\lambda_1} + \cdots + \alpha_p \delta_{\lambda_p}\| = |\alpha_1| + \cdots + |\alpha_p|.$$

Theorem 2.1. *Every filling subspace V of $\ell_\infty(\Lambda)$ verifies that the set of its norm-attaining functionals is* $\mathrm{card}(\mathrm{supp}(V))$*-lineable.*

Proof. Since $\mathrm{supp}(V)$ is infinite, we can decompose it as $\mathrm{supp}(V) = \dot{\bigcup}_{\lambda \in \mathrm{supp}(V)} A_\lambda$ with $\mathrm{card}(A_\lambda) = \aleph_0$ for all $\lambda \in \mathrm{supp}(V)$. By hypothesis, for every $\lambda \in \mathrm{supp}(V)$, there exists $x_\lambda \in \mathsf{S}_V$ such that $\mathrm{supp}(x_\lambda) \subseteq A_\lambda$ and there exists $\gamma_\lambda \in A_\lambda$ with $|x_\lambda(\gamma_\lambda)| = 1$. We will show now that $\mathrm{span}\{\delta_{\gamma_\lambda} : \lambda \in \mathrm{supp}(V)\} \subseteq \mathsf{NA}(V)$. Indeed, let $\lambda_1, \dots, \lambda_p \in \mathrm{supp}(V)$ all different and $\alpha_1, \dots, \alpha_p \in \mathbb{K}$. By keeping in mind that the A_λ's are all disjoint, note that

$$\left\|\mathrm{sgn}(\alpha_1) x_{\lambda_1} + \cdots + \mathrm{sgn}(\alpha_p) x_{\lambda_p}\right\|_\infty = 1$$