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Uniform approximation in the spherical distance by functions meromorphic on Riemann surfaces ☆

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Dedicated to Professor Richard Aron with admiration

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ABSTRACT

Given a function $f : E \to \overline{\mathbb{C}}$ from a closed subset of a Riemann surface R to the Riemann sphere $\overline{\mathbb{C}}$, we seek to approximate f in the spherical distance by functions meromorphic on R. As a consequence we generalize a recent extension of Mergelyan's theorem, due to Fragoulopoulou, Nestoridis and Papadoperakis [12]. The problem of approximating by meromorphic functions pole-free on E is equivalent to that of approximating by meromorphic functions zero-free on E, which in turn is related to Voronin's spectacular universality theorem for the Riemann zeta-function.

Given a function f defined on a closed subset E of a Riemann surface R, and given $\epsilon > 0$, a natural question is whether there exists a function f_ϵ meromorphic on R, such that $d(f(p), f_\epsilon(p)) < \epsilon$, for all $p \in E$, for a given distance function d. If ϵ is an arbitrary constant, we are speaking of uniform approximation, whereas, if $\epsilon = \epsilon(p)$ is an arbitrary positive continuous function, we call this tangential approximation. The two most natural distance functions here are the Euclidean distance $|\cdot - \cdot|$ and the chordal distance $\chi(\cdot,\cdot)$. If a function has no poles (respectively zeros) on E, we say that it is pole-free (respectively zero-free) on E. If the approximating functions f_ϵ are pole-free (respectively zero-free) on E, we call this pole-free (respectively zero-free) approximation. Unless explicitly stated otherwise, all approximations will be by meromorphic functions (equivalently, holomorphic mappings $R \to \overline{\mathbb{C}}$). We shall consider eight types of meromorphic approximation:

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- pole-free Euclidean uniform (respectively tangential) approximation,
- spherically uniform (respectively tangential) approximation,
- pole-free spherically uniform (respectively tangential) approximation,
- zero-free Euclidean uniform (respectively tangential) approximation.

Our main concern is with pole-free spherically tangential approximation, but we also discuss the other types of approximation, to situate our investigation in the general context of meromorphic approximation.

Spherical rational approximation on compact subsets of the plane was studied in [21] by Roth, Walsh and the first author. That paper concluded with a promise and two open problems. The promise was to consider in a later paper, spherical *meromorphic* approximation on *closed* sets. One open problem was to replace subsets of the plane by subsets of a Riemann surface. The other open problem was to ask whether we can spherically approximate a function f on a subset E by meromorphic functions pole-free on E, provided f is pole-free on E^0. The present paper finally addresses this promise and these open problems with a 39 year delay.

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1. Pole-free Euclidean uniform and tangential approximation

Denote by $\overline{\mathbb{C}}$ the Riemann sphere $\mathbb{C} \cup \{\infty\}$. For a compact subset $K \subset \mathbb{C}$, the theory of rational approximation studies the class $R(K)$ of functions on K, which are uniform Euclidean limits on K of rational functions which are pole-free on K.

In this section we present extensions to Riemann surfaces of classical results on rational approximation. Some of these extensions are known, but since we shall need them in subsequent sections and since they are sometimes difficult to find in the literature and notations and terminology are different in various sources, we present them here using our notation and terminology, for the convenience of the reader.

If E is a closed subset of a Riemann surface R, we denote by $M(E)$ the space of functions $E \to \mathbb{C}$, which are Euclidean uniform limits of functions meromorphic on R, which are pole-free on E. The subject of uniform approximation by such pole-free functions has been studied extensively, although many problems remain open.

We recall (see for example [19]) that every open Riemann surface admits a Cauchy kernel $C(p,q)$, that is, a meromorphic function on $R \times R$, whose only singularities are poles along the diagonal, such that, for some holomorphic function H on $R \times R$ and every point $p_0 \in R$, there is a parametric disc $\rho : D_0 \to \Delta$, where Δ is the unit disc in $\mathbb{C}$, such that, writing $(\rho \times \rho)(p,q) = (z,w)$,

$$C(p,q) = \frac{1}{z-w} + H\big((\rho \times \rho)^{-1}(z,w)\big), \quad (p,q) \in D_0 \times D_0.$$

A Bordered Riemann surface R is a connected Hausdorff space S, with an open covering $\{U_i\}$ of S and corresponding homeomorphisms h_i, $h_i : U_i \to V_i = h_i(U_i)$ where V_i is a relatively open subset of the closed upper half-plane. Also whenever $U_i \cap U_j \neq \emptyset$, the composition $h_i \circ h_j^{-1}$ is a conformal homeomorphism of the open set $h_j(U_i \cap U_j)$ on to $h_i(U_i \cap U_j)$.

Definition 1.1. A subset C of a Riemann surface R is called a 1-dimensional (real) sub-manifold if every $p \in C$ has an open neighbourhood N which can be mapped homeomorphically onto $|z| < 1$ such that the intersection $N \cap C$ corresponds to the real interval $(-1,+1)$.

Definition 1.2. A region Ω in a Riemann surface R is regularly embedded if $\partial\Omega$ is a 1-dimensional sub-manifold and $\partial\Omega = \partial(R \setminus \overline{\Omega})$ and in this case we say that $\overline{\Omega}$ is a closed regularly embedded region. It is easy to see (cf. [1]) that $\overline{\Omega}$ is a bordered surface whose border $b\Omega$ coincides with its boundary $\partial\Omega$. By a Jordan