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Ellipses and compositions of finite Blaschke products

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ABSTRACT

We provide a new proof of a theorem of Fujimura characterizing Blaschke products of degree-4 that are compositions of two degree-2 Blaschke products, connect this result to the numerical ranges of certain operators, and characterize Poncelet ellipses that are inscribed in quadrilaterals geometrically and algebraically.

1. Introduction

Poncelet's closure theorem is a well-known geometric result concerning polygons that are inscribed in an ellipse and circumscribe a smaller ellipse, [3]. Ellipses that can be so circumscribed are called *Poncelet ellipses*. Although the theorem holds for both convex and star polygons, in what follows we assume that all polygons are convex. *Finite Blaschke products*, on the other hand, are analytic functions of a complex variable of the form $B(z) = e^{i\theta}\prod_{j=1}^{N}\frac{z-\alpha_j}{1-\overline{\alpha_j}z}$, with $\theta \in \mathbb{R}$ and $|\alpha_j| < 1$. These are precisely the functions that are analytic on an open set containing the closed unit disk, map the unit disk to itself and the unit circle to itself. It turns out there is a deep connection between Blaschke products and Poncelet ellipses, [1] and [8]: Given a Blaschke product of degree 3 mapping 0 to 0, say $B_1(z) = zB(z)$ where B is degree 2, the solutions to the equation $B_1(z) = \lambda$ where $|\lambda| = 1$ form the vertices of triangles inscribed in the unit circle. The theorem, presented in [1], establishes that as λ runs over the entire unit circle, $\partial\mathbb{D}$, all triangles formed in this manner circumscribe a single ellipse, called the associated Poncelet curve to the Blaschke product B_1. We call these Blaschke 3-ellipses. These are, in particular, *Poncelet 3-ellipses*, because they are inscribed in triangles and have the Poncelet property. (As we shall see, an ellipse is a Blaschke 3-ellipse if

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and only if it is a Poncelet 3-ellipse, a fact that is true for higher degree as well, [4].) Moreover, the foci of this Poncelet ellipse coincide with the two zeros of B. The precise equation for the ellipse is given in [1]: A degree-3 Blaschke product B_1 with zeros 0, a, b admits an associated Poncelet ellipse with equation

$$|z-a|+|z-b|=|1-\bar{a}b|.$$

Fujimura extended this result to degree-4 Blaschke products giving necessary and sufficient conditions for a quadrilateral inscribed in $\partial\mathbb{D}$ to circumscribe an ellipse [5]. In doing so, Fujimura extended the connection of Poncelet ellipses to degree-4 Blaschke products that are *decomposable*; that is, can be written as the composition of two non-trivial Blaschke products.

Poncelet curves and Blaschke products are also related to a certain class of square matrices. A complex $n \times n$ matrix A is said to be in the class $\mathcal{S}_n$ if A is a contraction, has no eigenvalues of modulus 1, and satisfies $\operatorname{rank}(I - A^*A) = 1$. Gau and Wu have written extensively about the connections between numerical range, matrices in $\mathcal{S}_n$, and the Poncelet property; see [7] for a broad overview of the subject.

These matrices are interesting because they represent the compression of the shift operator to a finite-dimensional Hilbert space. It can be shown, up to unitary equivalence, that all matrices in $\mathcal{S}_n$ are representations of the compression of the shift operator, denoted S_B, to the model space $K_B = H^2 \ominus BH^2$, where H^2 denotes the Hardy space on the unit circle (see Section 2 for more information). The eigenvalues of the corresponding matrix A_B are the zeros of the Blaschke product B (see [7], pg. 180). Moreover, the numerical ranges of matrices in $\mathcal{S}_n$ are contained in $\mathbb{D}$ and possess the $(n+1)$-Poncelet property (see [13], Theorem 1); that is, there exists an infinite family of circumscribing $(n+1)$-gons inscribed in $\partial\mathbb{D}$ that are given by the boundaries of the numerical ranges of all unitary-1 dilations of the operator in $\mathcal{S}_n$. Furthermore, every point on $\partial\mathbb{D}$ is the vertex of such an $(n+1)$-gon. These curves are, therefore, called *Poncelet curves*. In [2], the authors pointed out that one can determine the associated Poncelet curve of a finite Blaschke product B by examining the matrix representing S_B. Thus, the numerical range of S_B, which we consider via its matrix A_B and denote it by $W(A_B)$, has the $(n+1)$-Poncelet property. We obtain $W(A_B)$ by intersecting the $(n+1)$-gons obtained with the points the degree-$(n+1)$ Blaschke product $B_1(z) = zB(z)$ sends to a point of the unit circle, see [7]. When we think of the curve, we may associate it with B and S_B or, when we consider the circumscribing $(n+1)$-gons we may associate it with $B_1(z) = zB(z)$.

In this paper, we provide a computational proof of Fujimura's theorem that says that a Poncelet curve inscribed in a quadrilateral and associated with a degree-4 Blaschke product $B_1(z) = zB(z)$ is an ellipse if and only if the Blaschke product is a composition of two degree-2 Blaschke products that map 0 to 0. This provides us with an apparently different formula for the ellipse, which we show is the same as that obtained by Fujimura. We put this result into its operator theoretic context by characterizing the numerical range of the compression of the shift operator S_B and are able to say more about the placement of the three zeros of B. In fact, the location of two of the three zeros of B completely determines the location of the third. Finally, we show that for any two points b and c in $\mathbb{D}$ there is a unique Poncelet 4-ellipse with foci at b and c, we provide a formula for this ellipse and we conclude by providing a few more geometric characterizations of Poncelet 4-ellipses in the spirit of Brianchon's classical theorem.

We note that this result on composition cannot be extended in a straightforward manner to Blaschke products of higher degree. For instance, $B_1(z) = z^5$ is a finite Blaschke product satisfying $B_1(0) = 0$ and its Poncelet curve is a circle: The solutions to $B_1(z) = \lambda$ are equally spaced on the unit circle, so they form the vertices of regular pentagons, all of which are rotates of each other and thus circumscribe a single circle. However, B_1 cannot be written as the composition of two non-trivial Blaschke products, since the degree of B_1 is prime. In Section 4 we show that there are degree-6 decomposable Blaschke products that are associated with Poncelet 6-ellipses (e.g., $B_1(z) = z^6$) and degree-6 decomposable Blaschke products that are not associated with ellipses. Thus, a simple characterization of which Blaschke products admit Poncelet ellipses for all degrees is, at least presently, out of reach.