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On sections of convex bodies in hyperbolic space ☆

K.L.H. Hiripitiyage, V. Yaskin *

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada

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ABSTRACT

We consider two problems on sections of convex bodies in hyperbolic space. The first one is a modified version of the Busemann–Petty problem. We look at conditions that guarantee a positive answer to this problem in all dimensions. The second problem is an analogue of a result of Makai, Martini, and Ódor about origin-symmetry. If in every direction the parallel section function has a critical value at zero, then the body is origin-symmetric. For both problems we use Fourier transform techniques.

1. Introduction

In 1956, Busemann and Petty [1] asked the following question. If $K, L \subset \mathbb{R}^n$ are origin-symmetric convex bodies such that

$$\mathrm{vol}_{n-1}(K \cap \xi^\perp) \le \mathrm{vol}_{n-1}(L \cap \xi^\perp), \qquad \forall\, \xi \in S^{n-1},$$

is it necessary that $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$? Here, $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi\rangle = 0\}$.

The Busemann–Petty problem was completely solved only in the nineties of the last century due to efforts of many mathematicians. The answer is affirmative if $n \le 4$ and negative if $n \ge 5$; see [6] for the history of the problem and its solution. Since the answer to the problem is negative in most dimensions, it is natural to look for modified conditions that make the answer positive in all dimensions. This was done by Koldobsky, Yaskin, and Yaskina [8].

Let $K \subset \mathbb{R}^n$ be a convex body. Its section function is defined by

$$S_K(\xi) = \mathrm{vol}_{n-1}(K \cap \xi^\perp), \quad \xi \in S^{n-1}.$$

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* Corresponding author.
E-mail addresses: hiripiti@ualberta.ca (K.L.H. Hiripitiyage), yaskin@ualberta.ca (V. Yaskin).

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We extend this function from the sphere to $\mathbb{R}^n \setminus \{0\}$ as a homogeneous function of degree -1. For $\alpha \in \mathbb{R}$, define the fractional Laplacian $(-\Delta)^{\alpha/2}$ of S_K by

$$(-\Delta)^{\alpha/2} S_K = \frac{1}{(2\pi)^n} \left(|x|_2^\alpha \widehat{S_K}(x) \right)^\wedge,$$

with the Fourier transform taken in the sense of distributions. It is proved in [8] that if $K, L \subset \mathbb{R}^n$ are infinitely smooth origin-symmetric convex bodies such that

$$(-\Delta)^{\alpha/2} S_K(\xi) \le (-\Delta)^{\alpha/2} S_L(\xi), \quad \forall \xi \in S^{n-1}, \tag{1}$$

for some $\alpha \in \mathbb{R}$ with $n-4 \le \alpha \le n-1$, then $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$. For $0 \le \alpha < n-4$, there are origin-symmetric convex bodies $K, L \subset \mathbb{R}^n$ such that equation (1) holds, but $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$. Observe that $\alpha = 0$ corresponds to the classical Busemann–Petty problem.

In recent years, the Busemann–Petty problem was considered in spaces other than real Euclidean; see [2,7,12]. In particular, the Busemann–Petty problem in hyperbolic and spherical spaces was solved in [12]. To formulate the result in hyperbolic space, let us use the following notation. Fix an origin O in $\mathbb{H}^n$ and denote by $T_O(\mathbb{H}^n)$ the tangent space to $\mathbb{H}^n$ at O. Consider the unit sphere S^{n-1} in $T_O(\mathbb{H}^n)$. For each $\xi \in S^{n-1}$ let $\xi^\perp$ denote the unique totally geodesic submanifold of $\mathbb{H}^n$ passing through O, whose normal vector at O is ξ. We will use hvol to denote volume in hyperbolic space. Let K and L be origin-symmetric convex bodies in hyperbolic space $\mathbb{H}^n$ such that

$$\mathrm{hvol}_{n-1}(K \cap \xi^\perp) \le \mathrm{hvol}_{n-1}(L \cap \xi^\perp), \qquad \forall\, \xi \in S^{n-1},$$

does it follow that $\mathrm{hvol}_n(K) \le \mathrm{hvol}_n(L)$? It was shown in [12] that this is true if $n = 2$, and false if $n \ge 3$. In view of this it is natural to modify the assumptions of the hyperbolic Busemann–Petty problem to obtain the affirmative answer in all dimensions, as it was done in the Euclidean case in [8]. As above we can define the hyperbolic section function

$$HS_K(\xi) = \mathrm{hvol}_{n-1}(K \cap \xi^\perp), \quad \xi \in S^{n-1},$$

extend it to $\mathbb{R}^n$ as a homogeneous function of degree -1, and consider $(-\Delta)^{\alpha/2} HS_K$.

We show that equation (1), interpreted in the setting of hyperbolic space, ensures $\mathrm{hvol}_n(K) \le \mathrm{hvol}_n(L)$ when $n-2 \le \alpha < n-1$. For $0 \le \alpha < n-2$, we find counterexamples. Our proof is based on the study of the Fourier transform of the distribution

$$\frac{|x|_2^{-\alpha} \|x\|_K^{-1}}{1 - \left(\frac{|x|_2}{\|x\|_K} \right)^2}.$$

We would like to note that this requires ideas different from those used in [8].

Let us now discuss the second problem that we study in this paper. Let K be a convex body in $\mathbb{R}^n$. Its parallel section function in the direction $\xi \in S^{n-1}$ is defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap \{\xi^\perp + t\xi\}), \quad t \in \mathbb{R}.$$

Brunn's theorem implies that if K is origin-symmetric and convex, then $\max_{t\in\mathbb{R}} A_{K,\xi}(t) = A_{K,\xi}(0)$, for every $\xi \in S^{n-1}$. A natural question is whether the converse is true and it was affirmatively answered by Makai, Martini, and Ódor [9]. If K is a convex body in $\mathbb{R}^n$ such that $A_{K,\xi}(0) = \max_{t\in\mathbb{R}} A_{K,\xi}(t)$ for all $\xi \in S^{n-1}$, then K is origin-symmetric.