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Affine invariant points and new constructions

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ABSTRACT

In [2] Grünbaum asked if the set of all affine invariant points of a given convex body is equal to the set of all points invariant under every affine automorphism of the body. In [3] we have proven the case of a body with no nontrivial affine automorphisms. After some partial results [6,7] the problem was solved in positive by Mordhorst [8]. In this note we provide an alternative proof of the affirmative answer, developing the ideas of [3]. Moreover, our approach allows us to construct a new large class of affine invariant points.

1. Introduction

Let $\mathbb{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$ and let $P : \mathbb{K}^n \to \mathbb{R}^n$ be a function satisfying the following two conditions:

1. For every nonsingular affine map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ and every convex body $K \in \mathbb{K}^n$ one has $P(\varphi(K)) = \varphi(P(K))$.
2. $P(K)$ is continuous in the Hausdorff metric.

Such a function P is called an *affine-invariant point*. The centroid and the center of the John ellipsoid (the ellipsoid of maximal volume contained in a given convex body) are examples of affine-invariant points [6].

Let $\mathcal{P}$ be the set of all affine-invariant points in $\mathbb{R}^n$. It was shown in [7] that $\mathcal{P}$ is an affine subspace of the space of continuous functions on $\mathbb{K}^n$ with values in $\mathbb{R}^n$. Grünbaum [2] asked a natural question: how big is the set $\mathcal{P}$? In particular, how to describe the set $\mathcal{P}(K) = \{P(K) \mid P \in \mathcal{P}\}$ for a given $K \in \mathbb{K}^n$? Denote the set of points fixed under affine maps of K onto itself by $\mathcal{F}(K)$. Grünbaum observed that $\mathcal{P}(K) \subset \mathcal{F}(K)$ and asked the following question:

Question 1.1. *Is the set $\mathcal{P}$ big enough to ensure that $\mathcal{P}(K) = \mathcal{F}(K)$ for every $K \in \mathbb{K}^n$?*

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In [7], Meyer, Schütt and Werner proved that the set of convex bodies K for which $\mathcal{P}(K) = \mathbb{R}^n$ is dense in $\mathbb{K}^n$. Then the author showed that if $\mathcal{F}(K) = \mathbb{R}^n$ then $\mathcal{P}(K) = \mathbb{R}^n$ [3]. Very recently, using a completely different approach, Mordhorst [8] has shown the affirmative answer to the Question 1.1. This proof used a previous development by P. Kuchment [4,5]. The purpose of this note is to show that the method of [3] can be also used to answer Question 1.1, providing a new proof. Moreover, we construct a new large class of affine invariant points.

2. Definitions and notation

Recall some basic notations from group theory.

The group of all invertible linear transformations of $\mathbb{R}^n$ is denoted by $GL(n, \mathbb{R})$. The group of all invertible linear transformations with the determinant equal to 1, i.e. the transformations which preserve volume and orientation is denoted by $SL(n, \mathbb{R})$.

For the purposes of the current paper we will use the group of all linear transformations preserving volume but not necessarily preserving orientation, i.e. the transformations with the determinant equal ± 1 denoted by SL_n^-.

The group of all affine transformations of $\mathbb{R}^n$ is denoted by $Aff(n)$. It may be represented as $GL(n) \ltimes \mathbb{R}^n$ with the rule $(r, x)(a) = r(a) + x$ where $r \in GL(n)$, $x, a \in \mathbb{R}^n$.

The unit Euclidian ball in $\mathbb{R}^n$ is denoted by B_2^n. The Euclidian norm of a vector is denoted by $|x|$. The Lebesgue measure on $\mathbb{R}^n$ is denoted by μ.

A right (left) Haar measure is a measure on a locally compact topological group that is preserved under multiplication by the elements of the group from the right (left). The Lebesgue measure is an example of a Haar measure on $\mathbb{R}^n$. Right and left Haar measures are unique up to multiplication however, not necessarily equal to each other. In this paper we always use a left Haar measure and denote the Haar measure of a set X by $\text{meas}(X)$.

$SAff(n)$ is the group of all affine transformations of $\mathbb{R}^n$ preserving volume. This group may be represented as a semidirect product of the group of all matrices with determinants equal to ± 1 and $\mathbb{R}^n$ with the rule $(r, x)(a) = r(a) + x$ for every $r \in GL(n)$ with $\det(r) = \pm 1$, $x \in \mathbb{R}^n$. $SAff(n)$ is equipped with the Haar measure, which is the product of Haar measures on the group of all matrices with the determinant equal to ± 1 and the group $\mathbb{R}^n$.

The Hausdorff metric is a metric on $\mathbb{K}^n$, defined as

$$d_H(K_1, K_2) = \min\{\lambda \geq 0 : K_1 \subset K_2 + \lambda B_2^n; K_2 \subset K_1 + \lambda B_2^n\}.$$

By $\mathbb{K}_1^n$ we denote the set of all convex compact sets in $\mathbb{R}^n$ with volume 1.

3. Affine invariant points

For a given convex body $K \in \mathbb{K}^n$ a family of affine invariant points is constructed by taking an arbitrary point v and averaging all possible affine transformations of this point with the weight

$$F = F_K : \mathbb{K}^n \to C(SAff(n))$$

defined by

$$F_K(L)(\varphi) = \mu(\varphi^{-1}(L) \cap K), L \in \mathbb{K}^n, \varphi \in SAff(x).$$