



Analytic cohomology groups of infinite dimensional complex manifolds [☆]



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ABSTRACT

Given a cohesive sheaf $\mathcal{S}$ over a complex Banach manifold M , we endow the cohomology groups $H^q(M, \mathcal{S})$ of M and $H^q(\mathfrak{U}, \mathcal{S})$ of open covers $\mathfrak{U}$ of M with a locally convex topology. Under certain assumptions we prove that the canonical map $H^q(\mathfrak{U}, \mathcal{S}) \rightarrow H^q(M, \mathcal{S})$ is an isomorphism of topological vector spaces.

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1. Introduction

Sheaf theory is an indispensable tool in the study of complex manifolds. For finite dimensional manifolds the relevant theory is that of coherent sheaves, and for Banach manifolds with Patyi we developed in [10] an analogous theory of cohesive sheaves. In [10] we primarily dealt with manifolds modeled on Banach spaces with an unconditional basis, but in light of Patyi's more recent work [15,16], the results of [10] hold more generally, e.g. for manifolds modeled on Banach spaces with a Schauder basis. The focus of [10] was to prove that higher cohomology groups of cohesive sheaves $\mathcal{S} \rightarrow M$ vanish, under a suitable convexity condition on the manifold M . Our goal here is to introduce a topology on the space $\Gamma(M, \mathcal{S})$ of sections and on the cohomology groups $H^q(M, \mathcal{S})$ of rather general manifolds, and to prove a topological version of Leray's isomorphism theorem. The topology on $H^q(M, \mathcal{S})$ is obtained as the direct limit of topologies on the Čech groups $H^q(\mathfrak{U}, \mathcal{S})$ corresponding to open covers $\mathfrak{U}$ of M , and our main result, Theorem 4.5, says that if the cohesive sheaf $\mathcal{S}$ is separated in a certain sense, and elements of $\mathfrak{U}$ have a suitable convexity property ($\mathfrak{U}$ is a “Stein” cover), then the canonical map $H^q(\mathfrak{U}, \mathcal{S}) \rightarrow H^q(M, \mathcal{S})$ is an isomorphism of topological vector spaces.

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In the finite dimensional theory the importance of topology on cohomology groups has long been recognized, to study finiteness, compactly supported cohomology groups, analytic continuation, and embedding problems [2,3,8,18–21]. By contrast, in infinite dimensions so far only $\Gamma(M, \mathcal{S})$ has been considered as a topological space, when $\mathcal{S}$ is the sheaf of germs of holomorphic functions valued in some topological vector space E , so that $\Gamma(M, \mathcal{S})$ can be identified with the space $\mathcal{O}^E(M)$ of holomorphic maps $M \rightarrow E$; see [6]. When $\dim M < \infty$, there is only one reasonable topology on the groups $\Gamma(M, \mathcal{S})$ and $H^q(M, \mathcal{S})$. Not so for infinite dimensional M . There are several equally reasonable topologies on the spaces $\mathcal{O}^E(M)$, and each gives rise to a topology on $H^q(M, \mathcal{S})$. However, there is only one among these topologies for which a topological Leray isomorphism could be proved. It is induced by the so-called τ_δ or countable cover topology on $\mathcal{O}^E(M)$, first studied by Coeuré and Nachbin in a somewhat lesser generality [4,13]. The advantage of this topology is that it is the direct limit of Banach space topologies.

Since the canonical map $H^q(\mathfrak{U}, \mathcal{S}) \rightarrow H^q(M, \mathcal{S})$ is always continuous, and under a suitable convexity condition on the cover $\mathfrak{U}$ it is bijective, to prove our main result we need to show that it is open as well. This will follow from two theorems. The first of these two is a very general theorem about sheaves $\mathcal{S}$ of Abelian groups over a topological space M , for which a topology is introduced, in a certain way, on the cochain and cohomology groups $C^q(\mathfrak{U}, \mathcal{S})$, $H^q(\mathfrak{U}, \mathcal{S})$ of any open cover $\mathfrak{U}$ of M . Let $\mathfrak{V}$ be another open cover, finer than $\mathfrak{U}$. If the Čech differential has a certain openness property, then by Theorem 2.2 the refinement homomorphisms $H^q(\mathfrak{U}, \mathcal{S}) \rightarrow H^q(\mathfrak{V}, \mathcal{S})$ are open. To apply this result to cohesive sheaves over complex manifolds, all one has to do is to prove the required property of the Čech differential in (infinite dimensional) Stein manifolds. This is the content of Theorem 4.6, whose proof rests on the vanishing theorem in [10].

We hope to use the results of this paper to study group actions and relative duality in infinite dimensional manifolds, and analytic continuation in mapping spaces.

In the paper we will freely use basic sheaf theory and complex analysis, for which good references are [1,6,12,17,22].

2. Cohomology groups of topologized sheaves

In this section M is a Hausdorff space and $\mathcal{S} \rightarrow M$ is a sheaf of Abelian groups. We will use standard notation of sheaf theory. If $\mathfrak{U} = \{U_i\}_{i \in I}$ is an open cover of M and $s = (i_0, \dots, i_q) \in I^{q+1}$ is a q -simplex, $q \geq 0$, we write $U_s = U_{i_0} \cap \dots \cap U_{i_q}$. We also introduce a (-1) -simplex $s = \emptyset$, which constitutes I^0 , and by $U_\emptyset$ we mean the whole space M . Our goal is to endow the group

$$C^q(\mathfrak{U}, \mathcal{S}) = \prod_{s \in I^{q+1}} \Gamma(U_s, \mathcal{S}), \quad q \geq -1,$$

of (not necessarily alternating) cochains with a topology. It would be natural to start by assuming the groups $\Gamma(U, \mathcal{S})$ are already endowed with a topology for open $U \subset M$, and give $C^q(\mathfrak{U}, \mathcal{S})$ the product topology. However, with ulterior motives we take a more general track, and assume that $\mathcal{S}$ is topologized in the following sense.

Definition 2.1. We say that the sheaf $\mathcal{S}$ is topologized if for any Hausdorff space U and any local homeomorphism $\pi: U \rightarrow M$ the group $\Gamma(U, \pi^{-1}\mathcal{S})$ is endowed with a topology, compatible with its group structure. It is required that if $\rho: V \rightarrow U$ is another local homeomorphism, the induced map

$$\rho^*: \Gamma(U, \pi^{-1}\mathcal{S}) \rightarrow \Gamma(V, (\pi\rho)^{-1}\mathcal{S})$$

be continuous.

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