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# Adjacency matrices of random digraphs: Singularity and anti-concentration

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A B S T R A C T

Let $\mathcal{D}_{n,d}$ be the set of all $d$-regular directed graphs on $n$ vertices. Let $G$ be a graph chosen uniformly at random from $\mathcal{D}_{n,d}$ and $M$ be its adjacency matrix. We show that $M$ is invertible with probability at least $1-C\ln^3 d/\sqrt{d}$ for $C \le d \le cn/\ln^2 n$, where $c, C$ are positive absolute constants. To this end, we establish a few properties of $d$-regular directed graphs. One of them, a Littlewood–Offord type anti-concentration property, is of independent interest. Let $J$ be a subset of vertices of $G$ with $|J| \approx n/d$. Let $\delta_i$ be the indicator of the event that the vertex $i$ is connected to $J$ and define $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \{0,1\}^n$. Then for every $v \in \{0,1\}^n$ the probability that $\delta = v$ is exponentially small. This property holds even if a part of the graph is "frozen."
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**Contents**



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## 1. Introduction

For $1 \leq d \leq n$ an undirected (resp., directed) graph $G$ is called *d-regular* if every vertex has exactly $d$ neighbors (resp., $d$ in-neighbors and $d$ out-neighbors). In this definition we allow graphs to have loops and, for directed graphs, opposite (anti-parallel) edges, but no multiple edges. Thus directed graphs (*digraphs*) can be viewed as bipartite graphs with both parts of size $n$. For a digraph $G$ with $n$ vertices its *adjacency matrix* $(\mu_{ij})_{i,j\leq n}$ is defined by

$$\mu_{ij} = \begin{cases} 1, & \text{if there is an edge from } i \text{ to } j; \\ 0, & \text{otherwise.} \end{cases}$$

For an undirected graph $G$ its adjacency matrix is defined in a similar way (in the latter case the matrix is symmetric). We denote the sets of all undirected (resp., directed) $d$-regular graphs by $\mathcal{G}_{n,d}$ and $\mathcal{D}_{n,d}$, respectively, and the corresponding sets of adjacency matrices by $\mathcal{S}_{n,d}$ and $\mathcal{M}_{n,d}$. Clearly $\mathcal{S}_{n,d} \subset \mathcal{M}_{n,d}$ and $\mathcal{M}_{n,d}$ coincides with the set of $n \times n$ matrices with 0/1-entries and such that every row and every column has exactly $d$ ones. By the probability on $\mathcal{G}_{n,d}$, $\mathcal{D}_{n,d}$, $\mathcal{S}_{n,d}$, and $\mathcal{M}_{n,d}$ we always mean the normalized counting measure.

Spectral properties of adjacency matrices of random $d$-regular graphs attracted considerable attention of researchers in the recent years. Among others, we refer the reader to [2,3,12,14,26,35] for results dealing with the eigenvalue distribution. At the same time, much less is known about the singular values of the matrices.

The present work is motivated by related general questions on singular probability. One problem was mentioned by Vu in his survey [37, Problem 8.4] (see also 2014 ICM talks by Frieze and Vu [15, Problem 7], [38, Conjecture 5.8]). It asks if for $3 \leq d \leq n-3$ the probability that a random matrix uniformly distributed on $\mathcal{S}_{n,d}$ is singular goes to zero as $n$ grows to infinity. Note that in the case $d = 1$ the matrix is a permutation matrix, hence non-singular; while in the case $d = 2$ the conjecture fails (see [37] and, for the directed case, [9]). Note also that $M \in \mathcal{M}_{n,d}$ is singular if and only if the "complementary" matrix $M' \in \mathcal{M}_{n,n-d}$ obtained by interchanging zeros and ones is singular, thus the cases $d = d_0$ and $d = n - d_0$ are essentially the same. The corresponding question for non-symmetric adjacency matrices is the following (cf., [9, Conjecture 1.5]):

*Is it true that for every* $3 \leq d \leq n-3$

$$p_{n,d} := \mathbb{P}_{\mathcal{M}_{n,d}}(\{M \in \mathcal{M}_{n,d} : M \textit{ is singular}\}) \longrightarrow 0 \quad \textit{as} \quad n \to \infty? \tag{$*$}$$

The main difficulty in such singularity questions stems from the restrictions on row- and column-sums, and from possible symmetry constraints for the entries. The question $(*)$ has been recently studied in [9] by Cook who obtained the bound $p_{n,d} \leq d^{-c}$ for a small universal constant $c > 0$ and $d$ satisfying $\omega(\ln^2 n) \leq d \leq n - \omega(\ln^2 n)$, where $f \geq \omega(a_n)$ means $f/a_n \to \infty$ as $n \to \infty$.

The main result of our paper is the following theorem.

**Theorem A.** *There are absolute positive constants* $c, C$ *such that for* $C \leq d \leq cn/\ln^2 n$ *one has*

$$p_{n,d} \leq \frac{C \ln^3 d}{\sqrt{d}}.$$

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