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Arc length as a global conformal parameter for analytic curves

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A R T I C L E I N F O

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A B S T R A C T

We show that arc length is a global conformal parameter for analytic curves and that this parameter can be used to decide whether the domain of definition of an analytic curve can be extended or not. The maximal extension with respect to the arc length parameter is the largest possible extension (over all parametrizations of the curve). Our proof is elementary, simple and short. Several examples are given in the plane, and the results remain true for curves in an arbitrary Euclidean space $\mathbb{R}^k$.

1. Introduction

In [2], the authors consider a domain Ω bounded by a finite set of disjoint analytic Jordan curves, each one parametrized in a way compatible with a Riemann map from the open unit disc in the interior of this Jordan curve. They prove that generically in $A^\infty(\Omega)$, every function f is nowhere (real) analytic on the boundary of Ω with respect to the above parameter. They also prove that for any Jordan analytic curve $\gamma : [0, 2\pi] \to \mathbb{C}$ with parameter t, a function f defined on the image of γ (and hence considered as a function of t) which is of class C^∞ (that is, $f \circ \gamma$ is a C^∞ function on $[0, 2\pi]$) is generically nowhere developable as a power series with variable t. A natural question which is addressed in the same paper is whether this set of C^∞ functions with respect to t coincides or not with the set of C^∞ functions with respect to arc length s, and whether the previous genericity result still holds with respect to s. This was the motivation for the present article. It turns out that the set of C^∞ functions is the same for any conformal parameter of the curve and since we show that arc length s is a global conformal parameter for analytic curves, it is clear

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that nothing changes in [2] if we replace t by s. In order to prove this, we are led to prove that arc length is a global conformal parameter for (not necessarily injective) analytic curves.

A curve $\gamma = \gamma(t)$ is said to be (real) analytic with respect to some real parameter t if this curve is locally representable as a power series in the parameter t and its derivative does not vanish at any point. Such a curve is also termed *regular analytic*, the term "regular" referring to the fact that the derivative is nonzero. We shall omit this adjective in what follows. We say that the parameter t for such a curve is a *conformal parameter*. We mention that any such curve is locally injective but globally it may not be. Information about real analytic functions can be found in [4].

If a curve is analytic with respect to some parameter t, then any two analytic extensions (with respect to this parameter) are compatible and thus, a maximal extension with respect to t exists. For instance, the maximal extension of the curve $\exp\frac{1}{t}$, $-1 < t < -\frac{1}{2}$ with respect to the parameter t is $\exp\frac{1}{t}$, $-\infty < t < 0$. The image of this curve is the segment $(0,1) \times \{0\} \subset \mathbb{C}$. It may appear at first sight that when t approaches 0^- or $-\infty$, the curve has two singularities. However, the segment $(0,1)$ may be continued to the whole real line $\mathbb{R}$ and the curve still be analytic, but with respect to some other (conformal) parameter; in particular with respect to the arc length parameter. This is a general fact: arc length can be used to decide whether an apparent singularity is essential or not; in the former case the curve cannot be extended. Also, the maximal extension of the curve with respect to the arc length parameter is the largest. We conclude that arc length is a global conformal parameter for any analytic curve.

The organization of the paper is as follows. Section 2 contains some preliminary results and definitions and ends with some observations on the maximal extension of an arbitrary planar analytic curve with respect to a specific parametrization. In Section 3, the main result is proven, that is, the fact that for an arbitrary planar analytic curve arc length is a global conformal parameter. The proof of this result is short and simple; a first undergraduate course in Complex Variables is sufficient to understand it. In Section 4, some examples are given where the maximal extension $\gamma^* : (A, B) \to \mathbb{C}$ of an analytic curve γ with respect to arc length is investigated. We give several examples where the limit sets at $s \to A^+$ or $s \to B^-$ of $\gamma^*(s)$ can be singletons or circles in $\mathbb{C} \cup \{\infty\}$.

After we circulated the first version of this paper [7], P. Gauthier informed us that for any connected compact subset L of $\mathbb{C} \cup \{\infty\}$ we can construct an analytic curve $\gamma : (A, B) \to \mathbb{C}$ such that L is the limit set at one endpoint A or B. We note that for a connected compact set K containing at least two points with empty interior and connected complement this result follows from the Riemann mapping theorem: Consider a conformal representation $\mathcal{D} : \mathbb{D} \to (\mathbb{C} \cup \{\infty\}) \setminus K$ of the complement of this set, and take any analytic curve Γ in the disc $\mathbb{D}$ whose limit set is the unit circle (e.g. a curve spiraling to the boundary of the disc; cf. Example 4.4 below). Then $\mathcal{D}(\Gamma)$ is an analytic curve whose limit set is K. If we do not assume that the interior of K is empty, then the limit set is the frontier of K. We also give an example (Example 4.10) where the limit set is a compact set with nonempty interior. Finally, in Section 5, we extend our result to analytic curves in $\mathbb{R}^k$.

2. Preliminaries

We start with the following well-known result, which is a particular case of a more general result on Riemann surfaces, cf. [3]. We include its proof for the purpose of completeness.

Lemma 2.1. *Let $a < b$ be two real numbers and $\Omega \subset \mathbb{C}$ an open set containing $[a, b]$. Let $\phi : \Omega \subset \mathbb{C}$ be a holomorphic function such that $\phi'(t) \neq 0$ for all $t \in [a, b]$ and $\phi|_{[a,b]}$ is one-to-one. Then there is a convex open set V such that $[a, b] \subset V \subset \Omega$ and $\phi|_V$ is one-to-one.*

Proof. For every $n = 1, 2, \ldots$ we consider the convex open set

$$V_n = \{z = x + iy \in \mathbb{C}\ ,\ a - \frac{1}{n} < x < b + \frac{1}{n}\ ,\ |y| < 1/n\}.$$