J. Math. Anal. Appl. 445 (2017) 1530–1547

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Upper bound for intermediate singular values of random matrices

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ARTICLE INFO

Article history:
Received 13 June 2016
Available online 11 August 2016
Submitted by D. Ryabogin

Dedicated to Richard Aron

Keywords:
Random matrix
Upper bound of singular value
Non-asymptotic distribution
Subgaussian

ABSTRACT

In this paper, we prove that an $n \times n$ matrix A with independent centered subgaussian entries satisfies

$$s_{n+1-l}(A) \leq C_1 t \frac{l}{\sqrt{n}}$$

with probability at least $1-\exp(-C_2 tl)$. This yields $s_{n+1-l}(A) \sim \frac{cl}{\sqrt{n}}$ in combination with a known lower bound. These results can be generalized to the rectangular matrix case.

1. Introduction

Consider an $n \times m$ real matrix A with $n \geq m$. The singular values $s_k(A)$ of A, where $k = 1, 2, \cdots, n$, are the eigenvalues of $\sqrt{A^T A}$ arranged in non-increasing order. The non-asymptotic singular value distribution of random i.i.d. sub-gaussian matrix is an important and interesting subfield in random matrix theory. The first result in this direction was obtained in [6], where it was proved that the smallest singular value of a square i.i.d. sub-gaussian matrix is bounded below by $n^{-3/2}$ with high probability. This result was later extended and improved in a number of papers, including [18,19,9,1,5]. The above mentioned results pertain to square matrices. A probabilistic lower bound for the smallest singular value of a rectangular matrix was obtained by M. Rudelson and R. Vershynin [10]. They proved that an $n \times (n-l)$ matrix has smallest singular value lower bounded by $\frac{\varepsilon l}{\sqrt{n}}$ with probability at least $1 - (C\varepsilon)^l - \exp(-Cn)$. Using this result, one can show that for a square i.i.d. sub-gaussian matrix A, $s_{n+1-l}(A) > c\frac{l}{\sqrt{n}}$ with high probability.

However, the optimal upper bound of the singular values for general sub-gaussian matrices is unknown. Prior to this paper, the only progress in this direction was made by Szarek [17] who proposed an optimal upper bound for the gaussian matrix. Szarek proved that for a standard gaussian i.i.d. matrix G, $\frac{Cl}{\sqrt{n}} \leq s_{n+1-l}(G) \leq \frac{Cl}{\sqrt{n}}$ with probability at least $1 - \exp(-Cl^2)$. This result suggests that lth smallest singular value of an i.i.d. sub-gaussian matrix is concentrated around $\frac{l}{\sqrt{n}}$.

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[1] Partially supported by M. Rudelson's NSF Grant DMS-1464514, and USAF Grant FA9550-14-1-0009.

http://dx.doi.org/10.1016/j.jmaa.2016.08.007

Although the optimal upper bound is not proved for general matrices, some results can be deduced. T. Tao and V. Vu have established the universal behavior of small singular values in [19] (see Theorem 6.2 [19]). Combined with Szarek's Theorem 1.3 in [17], their approach allows us to deduce some non-asymptotic bounds for random i.i.d. square matrix under a moment condition. However, their bound only works for $l \leq n^c$ where c is a small constant. Tao and Vu's approach [19] is based on the Berry–Esseen Theorem for the frames and does not allow one to obtain exponential bounds for the probability as we do in our Theorem 1.6. Also, C. Cacciapuoti, A. Maltsev, B. Schlein estimated the rate of convergence of the empirical measure of singular values to the limit distribution near the hard edge (see [2] Theorem 3). Theorem 3 in [2] can be used to derive an upper bound of the form $\frac{cl^C}{\sqrt{n}}$ [2]. Better understood is the upper bound for the smallest singular value. M. Rudelson and R. Vershynin were the first to prove the smallest singular value of the i.i.d. sub-gaussian matrix is also bounded from above by $\frac{c}{\sqrt{n}}$ with high probability (see [8]). A different proof with an exponential tail probability can be found in a very recent paper by H. Nguyen and V. Vu [4].

In this paper, we prove the upper bound on the singular values under two assumptions: that the entries of the matrix are non-degenerate; and that they have a fast tail decay. The first assumption is quantified in terms of the Levy concentration function and the second is quantified in terms of the ψ_θ-norm. Next we provide definitions.

Definition 1.1. Let Z be a random vector that takes values in $\mathbb{C}^n$. The concentration function of Z is defined as

$$\mathcal{L}(Z,t) = \sup_{u\in\mathbb{C}^n} \mathbb{P}\{\|Z-u\|_2 \leq t\}, t \geq 0.$$

Definition 1.2. Let $\theta > 0$. Let Z be a random variable on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then the ψ_θ-norm of Z is defined as

$$\|Z\|_{\psi_\theta} := \inf\left\{\lambda > 0 : \mathbb{E}\exp\left(\frac{|Z|}{\lambda}\right)^\theta \leq 2\right\}$$

If $\|X\|_{\psi_\theta} < \infty$, then X is called a ψ_θ random variable. This condition is satisfied for broad classes of random variables. In particular, a bounded random variable is ψ_θ for any $\theta > 0$, a normal random variable is ψ_2, and a Poisson variable is ψ_1.

In this paper, we prove that for all l, $s_{n+1-l}(A) \geq \frac{Ctl}{\sqrt{n}}$ with an exponentially small probability, where A is a random matrix under the following assumption:

Assumption 1.3. Let $p > 0$. Let A be an $n \times m$ random matrix with i.i.d. entries that have mean 0, variance 1 and ψ_2-norm K. Assume also that there exists $0 < s \leq s_0(p, K)$ such that

$$\mathcal{L}(A_{i,j}, s) \leq ps.$$

Here, $s_0(p, K)$ is a given function depending only on p and K.

A concrete value of $s_0(p, K)$ can be determined by tracing the proof of Theorem 1.6.

Remark 1.4. The condition on the Levy concentration function is automatically satisfied if the density of the entries is bounded by p. However, our result holds in a much more general setting because we require this condition to hold only for one fixed s and not for all $s > 0$. This assumption can be viewed as a discrete analog of the bounded density condition.