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Inverse problems for the differential operator on a graph with cycles

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A R T I C L E I N F O

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A B S T R A C T

This work deals with inverse problems of the Sturm–Liouville operator on a graph with multi-cycles, endowed with standard matching conditions in the internal vertex and with Neumann boundary conditions at the boundary vertices. We show that the potential on a graph with multi-cycles (including the potential on cycles) can be constructed by the dense nodal points on the interval considered. Moreover, we investigate the so-called incomplete inverse problems of recovering the potential on a fixed edge from a subset of nodal points situated only on a part of the edge.

1. Introduction

Consider a compact graph G in $\mathbf{R}^m$ with the set of vertices $V = \{v_0, \ldots, v_r\}$ and the set of edges $\mathcal{E} = \{e_1, \ldots, e_r, e_{r+1}, \ldots, e_{r+r_1}\}$, where $v_1, \ldots, v_r$ are the boundary vertices, v_0 is the internal vertex, $e_j = [v_j, v_0]$, $j = \overline{1, r}$, $\bigcap_{j=1}^{r+r_1} e_j = \{v_0\}$, and $e_{r+1}, \ldots, e_{r+r_1}$ are cycles. Thus, the graph G has r_1 cycles $e_{r+1}, \ldots, e_{r+r_1}$ and one internal vertex v_0 (see Fig. 1). We suppose that the length of each edge is equal to 1. Each edge $e_j \in \mathcal{E}$ is parameterized by the parameter $x \in [0, 1]$; below we identify the value x of the parameter with the corresponding point on the edge. It is convenient for us to choose the following orientation: for $j = \overline{1, r}$, the vertex v_j corresponds to $x = 0$, and the vertex v_0 corresponds to $x = 1$; for $j = \overline{r+1, r+r_1}$, both ends $x = +0$ and $x = 1 - 0$ correspond to v_0 (see Fig. 2).

An integrable function Y on G may be represented as $Y = \{y_j\}_{j=\overline{1,r+r_1}}$, where the function $y_j(x), x \in [0, 1]$, is defined on the edge e_j. Let $q = \{q_j\}_{j=\overline{1,r+r_1}}$ be an integrable real-valued function on G; q is called the potential. Consider the following differential equations on G:

$$-y_j''(x) + q_j(x)y_j(x) = \lambda y_j(x), \quad j = \overline{1, r+r_1}, \tag{1.1}$$

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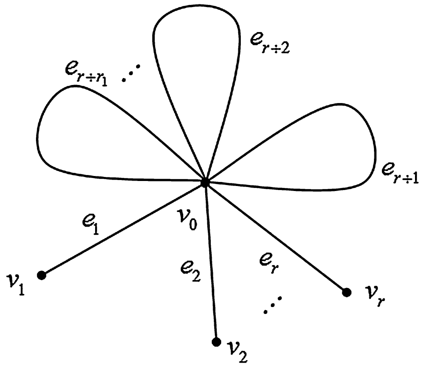

Fig. 1. Graph with multi-cycles.

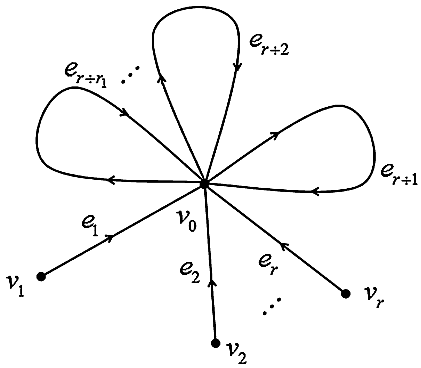

Fig. 2. Oriented graph with multi-cycles.

where λ is the spectral parameter, the functions $y_j, y'_j, j = \overline{1, r+r_1}$, are absolutely continuous on $[0, 1]$ and satisfy the following matching conditions in the internal vertex v_0:

$$\begin{cases} y_j(1) = y_i(0) & \text{(continuity condition)} \\ \sum_{j=1}^{r+r_1} y'_j(1) = \sum_{i=r+1}^{r+r_1} y'_i(0) & \text{(Kirchhoff's condition),} \end{cases} \tag{1.2}$$

where $j = \overline{1, r+r_1}, i = \overline{r+1, r+r_1}$. The matching conditions (1.2) are called the *standard conditions*. In electrical circuits, (1.2) expresses Kirchhoff's law; in elastic string networks, it expresses the balance of tension, and so on.

Let us consider the boundary value problem $B := B(q)$ on G for equations (1.1) with the matching conditions (1.2) and with Neumann boundary conditions at the boundary vertices $v_1, \dots, v_r$:

$$y'_j(0) - h_j y_j(0) = 0, \quad j = \overline{1, r}, \quad h_j \in \mathbb{R}. \tag{1.3}$$

In Section 2 we discuss a direct problem: asymptotics of eigenvalues and the oscillation of components of eigenfunctions for the problem B. Section 3 deals with the inverse nodal problem of recovering the potential from any dense subset of the nodal points of the problem B. The uniqueness theorem and a constructive procedure for the solution are provided. In Section 4 we investigate the so-called incomplete inverse problems of recovering the potential on a fixed edge from a subset of nodal points situated only on a part of the edge.

A quantum graph is a metric graph, equipped with a differential operator that acts as $-\frac{d^2}{dx^2} + q$ on each bond, together with boundary conditions at the vertices chosen to make the operator self-adjoint (see [2,8–10] and references therein). Inverse problems for differential operators on graphs appear frequently in mathematics, natural sciences and engineering. In particular, direct and inverse problems for such operators