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Estimates on derivatives and logarithmic derivatives of holomorphic functions and Picard's theorem

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ABSTRACT

This paper gives, in an elementary way, estimates on derivatives and logarithmic derivatives of holomorphic functions and then, as an application, a brief proof of Picard's theorem.

1. Estimates for holomorphic functions play an important role in complex analysis and its applications. The classic Borel–Carathéodory theorem shows that a holomorphic function and its derivatives of any order may be bounded by its real part (see e.g. [11], p. 176 and [6], p. 50). In this note, we first give an estimate on the average $\frac{1}{2\pi}\int_0^{2\pi}|f^{(n)}(re^{i\theta})|d\theta$ by the real part of a holomorphic function f with a short and elementary proof via Cauchy's theorem and formula only, without invoking any other theorems in complex analysis. This is partially motivated by the recent work [9], where elementary properties of entire functions were used to prove some old and new results on polynomials and entire functions, and by [8] on the so-called logarithmic derivative lemma (see below).

Theorem 1. *Let f be a holomorphic function in $|z| \le R$. Then for $0 < r < R$ and $n \ge 1$,*

$$\frac{1}{2\pi}\int_0^{2\pi}|f^{(n)}(re^{i\theta})|d\theta \le \frac{2n!}{(R-r)^n}\big(A(R,f) - Re\{f(0)\}\big), \tag{1}$$

where $A(r,f) = \max_{|z|\le r}\{Re\{f(z)\}\}$.

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Theorem 1 is naturally connected to the logarithmic derivative lemma concerning the average $\frac{1}{2\pi}\int_0^{2\pi}\log^+|\frac{f'(re^{i\theta})}{f(re^{i\theta})}|d\theta$, where $\log^+|x| = \max\{\log|x|, 0\}$. This is a central lemma in Nevanlinna's theory (see e.g. [5], p. 36 and [10], p. 241 for the general logarithmic derivative lemma in the plane and [8] for a version of the lemma in several complex variables). A simple application of Theorem 1 yields the following

Theorem 2. *Let f be a holomorphic function without zeros in $|z| \le R$. Then for $0 < r < R$,*

$$\frac{1}{2\pi}\int_0^{2\pi}|\frac{f'(re^{i\theta})}{f(re^{i\theta})}|d\theta \le \frac{2}{R-r}\big(\log M(R,f) - \log|f(0)|\big), \tag{2}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|\frac{f'(re^{i\theta})}{f(re^{i\theta})}|d\theta \tag{3}$$
$$\le \log^+\frac{1}{R-r} + \log^+\log^+ M(R,f) + \log^+\log^+\frac{1}{|f(0)|} + 3\log 2,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|\frac{f'(re^{i\theta})}{f(re^{i\theta})}|d\theta \tag{4}$$
$$\le 2\log^+\frac{1}{R-r} + \log^+ R + \log^+ m(R,f) + \log^+\log^+\frac{1}{|f(0)|} + 5\log 2,$$

where $M(r,f) = \max_{|z|\le r}\{|f(z)|\}$ and $m(r,f) = \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta}|d\theta$.

Holomorphic functions without zeros appear frequently in complex analysis and applications, for example, in questions on Picard type theorems, normal families, and complex differential equations. The proof of the general form of the logarithmic derivative lemma for meromorphic functions is very involved and is a necessary part of the general Nevanlinna theory (see e.g. [5] and [10]). Theorem 2 serves as this lemma for holomorphic functions without zeros with a simple and elementary proof suitable for a general audience and for situations where the full force of Nevanlinna theory is not necessary.

We note that the estimate (2) implies (3), and (3) implies (4); each might be of its own interest. (The estimate (4) is of the same form as that of the logarithmic derivative lemma with slightly better coefficients and without the term $\log^+\frac{1}{r}$, see [5], p. 36.) Each term (except possibly the constant term) on the right hand sides of these inequalities could be significant in making estimates (cf. [8]).

As an application, we give a proof of the famous Picard theorem.

Theorem 3 *(Picard's theorem). An entire function f omitting two distinct complex numbers must be constant.*

Picard's theorem is among the most striking results in complex analysis and a remarkable strengthening of Liouville's Theorem which states that a bounded entire function must be constant. While Liouville's Theorem can be treated as a consequence of Cauchy's formula (or a consequence of Cauchy's theorem directly, cf. [9]), Picard's theorem is generally not encountered until advanced complex analysis involving rather heavy machinery. Many proofs of Picard's theorem have been given (see [1–5,7,12], etc.). The proof here is brief and elementary, although the underlying ideas may be traced to the relevant techniques developed in Nevanlinna's theory (see [5]).

It is hoped that the elementary and brief proofs of this paper will be of some interest and the estimates may have further applications.