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Hardy–Rellich inequalities in domains of the Euclidean space

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ABSTRACT

For test functions supported in a domain of the Euclidean space we consider the Hardy–Rellich inequality: $\int |\Delta f|^2\,dx \geq C_2 \int |f|^2 \delta^{-4}(x)\,dx$, where $C_2 = const \geq 0$ and $\delta(x)$ is the distance from x to the boundary of the domain. M.P. Owen proved that this inequality is valid in any convex domain with $C_2 = 9/16$ (M.P. Owen (1999) [21]). We examine the inequality in non-convex domains. It is proved that a positive constant C_2 for a plane domain exists if and only if its boundary is a uniformly perfect set. For a domain of dimension $d \geq 2$ we prove that the inequality holds with the sharp constant $C_2 = 9/16$, if the domain satisfies an exterior sphere condition with certain restriction on the radius of the sphere. In addition, we obtain similar results for the inequality $\int \delta^2(x)|\Delta f|^2\,dx \geq C_2^* \int |f|^2 \delta^{-2}(x)\,dx$.

1. Introduction

In 1954 F. Rellich (see [22]) proved the following inequality

$$\int_{\mathrm{R}^d} |\Delta f|^2\,dx \geq 2^{-4} d^2 (d-4)^2 \int_{\mathrm{R}^d} |f|^2 |x|^{-4} dx \qquad \forall f \in C_0^\infty(\mathrm{R}^d \setminus \{0\}) \tag{1}$$

in the Euclidean space R^d for $d \geq 1$, $d \neq 2$, where Δf is the Laplacian of the test function $f : \mathrm{R}^d \setminus \{0\} \to \mathrm{C}$. In addition, Rellich showed that (1) is not valid for $d = 2$, even if one replaces the corresponding constant 1 by an arbitrarily small constant $\varepsilon > 0$. There are many papers by U.W. Schmincke, W. Allegretto, D.M. Bennett, E.B. Davies and A.M. Hinz, E. Mitidieri and other mathematicians (see, for instance, [23,1,13,16,20,14] and the literature therein), where one can find direct generalizations of (1).

Much less is known about inequalities of type (1) in domains $\Omega \subset \mathrm{R}^d$ with weight functions depending on other geometric quantities, different from $|x|$. In this paper we will study the case when weight functions depend on the distance function $\mathrm{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|,\ x \in \Omega$. More precisely, we are concerned with the following analogue of (1) due to M.P. Owen [21] (see, also, [12,11,17] for the same inequality with

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remainders): *Suppose that* $d \geq 1$ *and* Ω *is a convex domain in the Euclidean space* R^d, $\Omega \neq \mathrm{R}^d$. *Then one has the following inequality*

$$\int_\Omega |\Delta f|^2 \, dx \geq (9/16) \int_\Omega |f|^2 \mathrm{dist}^{-4}(x, \partial\Omega) dx \qquad \forall f \in C_0^\infty(\Omega). \tag{2}$$

The constant $9/16$ *is optimal since it is sharp for the half-space* $x_1 > 0$. Our aim is to study inequalities of type (2) in non-convex domains. More precisely, in an arbitrary domain $\Omega \subset \mathrm{R}^d$ we will consider the inequality

$$\int_\Omega |\Delta f|^2 \, dx \geq C_2(\Omega) \int_\Omega |f|^2 \mathrm{dist}^{-4}(x, \partial\Omega) dx \qquad \forall f \in C_0^\infty(\Omega) \tag{3}$$

with $C_2(\Omega)$ chosen as the maximal constant admissible at this place, i.e.

$$C_2(\Omega) := \inf_{f \in C_0^\infty(\Omega), f \not\equiv 0} \frac{\int_\Omega |\Delta f|^2 \, dx}{\int_\Omega |f|^2 \, \mathrm{dist}^{-4}(x, \partial\Omega) \, dx} \in [0, \infty).$$

The main aim of this paper is to study two natural problems related to inequality (3): a) *find a criterion of positivity of the constant* $C_2(\Omega)$ *for two-dimensional domains*; b) *describe domains* $\Omega \subset \mathrm{R}^d$ $(d \geq 2)$, *for which* $C_2(\Omega) = 9/16$. In addition, we will examine similar problems for the following inequality

$$\int_\Omega \mathrm{dist}^2(x, \partial\Omega) |\Delta f|^2 \, dx \geq C_2^*(\Omega) \int_\Omega |f|^2 \mathrm{dist}^{-2}(x, \partial\Omega) dx \quad \forall f \in C_0^\infty(\Omega), \tag{4}$$

where $\Omega \subset \mathrm{R}^d$ is a domain and the constant $C_2^*(\Omega) \in [0, \infty)$ is chosen to be maximal at this place. It is to note that the main results of this paper are announced without proof in the short communication [5].

2. Criterion of positivity of $C_2(\Omega)$ for $\Omega \subset \mathrm{R}^2$

Let $\Omega \subset \mathrm{R}^2$ be a domain such that $\Omega \neq \mathrm{R}^2$. We do not suppose that Ω is bounded, and consequently, it is possible that the boundary set $\partial\Omega$ contains the point at infinity. By $\overline{\mathrm{R}}^2$ we will denote the extended plane (the Riemann sphere). Following L. Carleson and T.W. Gamelin [15], we say that the boundary $\partial\Omega$ is **a uniformly perfect set** in $\overline{\mathrm{R}}^2$, if the maximal conformal modulus $M(\Omega)$ is finite. Recall that $M(\Omega)$ is defined as follows: *a)* $M(\Omega) = 0$, *if* Ω *is a simply connected domain; b)* $M(\Omega)$ *is the conformal modulus of* Ω, *if* Ω *is a doubly connected domain; c) if* $\partial\Omega$ *has more than two components, then* $M(\Omega)$ *is the supremum of the conformal moduli* $M(\Omega')$, *where* $\Omega' \subset \Omega$ *are doubly connected domains, separating* $\partial\Omega$.

There are many other definitions and characterizations of uniformly perfect sets. Moreover, there is a connection with Hardy's inequalities. Namely, let $\Omega \subset \mathrm{R}^2$ be a domain such that $\Omega \neq \mathrm{R}^2$, and let $c_2(\Omega) \in [0, \infty)$ be the maximal constant in the Hardy inequality

$$\int_\Omega |\nabla u|^2 \, dx \geq c_2(\Omega) \int_\Omega |u|^2 \mathrm{dist}^{-2}(x, \partial\Omega) dx \qquad \forall u \in C_0^\infty(\Omega). \tag{5}$$

It is known that *the constant* $c_2(\Omega) > 0$ *if and only if* $\partial\Omega$ *is a uniformly perfect set* (see, for instance, [2,24,3] and [18]).

We will also deal with the Euclidean maximal modulus $M_0(\Omega)$, defined by $M_0(\Omega) = \sup M(A)$, $M(A) = (2\pi)^{-1} \ln(r_2(A)/r_1(A))$, where the supremum is taken over all annuli