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On the solvability of asymptotically linear systems at resonance

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This paper is concerned with the solvability of the system

$$\begin{gathered} -\Delta u - \nu_1\theta_1 v = f(x,u,v) + h_1(x) \quad \text{in } \Omega\,; \\ -\Delta v - \nu_1\theta_2 u = g(x,u,v) + h_2(x) \quad \text{in } \Omega\,; \\ u = v = 0 \quad \text{on } \partial\Omega\,, \end{gathered}$$

at resonance at the simple eigenvalue ν_1 of the corresponding linear eigenvalue problem. Here $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with $C^{2,\eta}$-boundary $\partial\Omega$, $\eta \in (0,1)$ (a bounded interval if $N = 1$) and θ_1, θ_2 are positive constants. The nonlinear perturbations $f(x,u,v), g(x,u,v) : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions that are sublinear at infinity. We employ the Lyapunov–Schmidt method to provide sufficient conditions on $h_1, h_2 \in L^r(\Omega)$; $r > N$, to guarantee the solvability of the system.

1. Introduction

The purpose of this article is to use the Lyapunov–Schmidt method to discuss the solvability of the nonlinear elliptic system

$$\begin{gathered} -\Delta u - \nu_1\theta_1 v = f(x,u,v) + h_1(x) \quad \text{in } \Omega\,; \\ -\Delta v - \nu_1\theta_2 u = g(x,u,v) + h_2(x) \quad \text{in } \Omega\,; \\ u = v = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

at resonance. Here $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with $C^{2,\eta}$-boundary $\partial\Omega$, $\eta \in (0,1)$ (a bounded interval if $N = 1$), $h_1, h_2 \in L^r(\Omega)$ with $r > N$ and θ_1, θ_2 are positive constants. The fixed parameter

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$\nu_1 \overset{\text{def}}{=} \frac{\mu_1}{\sqrt{\theta_1\theta_2}} > 0$ is the simple eigenvalue of the linear eigenvalue problem

$$\begin{aligned} -\Delta u - \nu_1\theta_1 v = 0 \quad &\text{in } \Omega\,; \\ -\Delta v - \nu_1\theta_2 u = 0 \quad &\text{in } \Omega\,; \\ u = v = 0 \quad &\text{on } \partial\Omega\,. \end{aligned}$$

The eigenfunction corresponding to ν_1 is componentwise positive and is given by $(\sqrt{\theta_1}\varphi_1, \sqrt{\theta_2}\varphi_1)$ (see [11, Prop. B.1]), where (μ_1, φ_1) is the principal eigenpair of

$$\begin{aligned} -\Delta\varphi = \lambda\varphi \quad &\text{in } \Omega\,; \\ \varphi = 0 \quad &\text{on } \partial\Omega\,, \end{aligned}$$

normalized such that $\varphi_1 > 0$ in Ω and $(\theta_1 + \theta_2)\int\limits_\Omega \varphi_1^2\,\mathrm{d}x = 1$. The nonlinear perturbations $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions satisfying the following conditions:

(H1) $|f(x,u,v)|, |g(x,u,v)| \le b(x) + k(|u| + |v|)$ for some $b \in L^r(\Omega)$, $r > N$, $b(x) \ge 0$ a.e. in Ω, and $k > 0$, and

(H2) $\displaystyle\lim_{|u|+|v|\to+\infty} \frac{f(x,u,v)}{|u|+|v|} = \lim_{|u|+|v|\to+\infty} \frac{g(x,u,v)}{|u|+|v|} = 0$ for a.e. $x \in \Omega$.

The Lyapunov–Schmidt method, also called the Cesari alternative method, was successfully applied to variety of resonant problems of (mostly scalar) elliptic equations; starting with Landesman–Lazer type nonlinearity treated in the seminal paper [18], nonlinearity vanishing at infinity see, e.g., [2,4,5,12,20], and unbounded sublinear nonlinearity see, e.g., [19]. For survey, systematic treatment and application of this method to boundary value problems, see [6,7,10,14] and references therein.

Recently, study of systems of elliptic equations has gained greater interest due to their wider applicability to mathematical modeling of real world phenomena. However, initial results for systems at resonance trace back to late 1970s, see [7].

In a series of papers, [21–23], technique using the Lyapunov–Schmidt method was developed to treat elliptic system at resonance. However, smoothness assumption on the nonlinear perturbations was crucial in establishing the existence results. Using the Lyapunov–Schmidt method again, resonant elliptic systems that are coupled only in the nonlinear part were treated in [8,9]. Very recently in [16], implicit function theorem combined with continuation technique was used to establish the existence result for resonant elliptic system under Landesman–Lazer type conditions and an additional smoothness assumption. The smoothness hypothesis was relaxed in [11] by the authors using bifurcation theory. Finally, let us mention [17], where the classical result in [13] for the scalar equation was extended to the system.

Here we aim to present an approach using the Lyapunov–Schmidt method to study elliptic systems at resonance with nonlinear perturbations f and g that have been considered for the scalar case. In particular, we consider nonlinearities that are vanishing at infinity, unbounded but sublinear at infinity and those satisfying Landesman–Lazer type conditions for systems. Our approach allows us to treat systems that do not have variational structure and are weakly coupled in linear as well as nonlinear part. Moreover, we do not assume any smoothness conditions on the nonlinear perturbations.

We state our results in Section 2 and prove them in Section 3. In Section 4, we provide several examples satisfying the hypotheses of our theorems. In Appendix A, we describe the abstract Lyapunov–Schmidt method in the Banach space for the convenience of readers. In Appendix B, we prove an auxiliary lemma essential to prove our theorems.