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On a representation theorem for finitely exchangeable random vectors

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ABSTRACT

A random vector $X = (X_1, \ldots, X_n)$ with the X_i taking values in an arbitrary measurable space $(S, \mathscr{S})$ is exchangeable if its law is the same as that of $(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ for any permutation σ. We give an alternative and shorter proof of the representation result (Jaynes [6] and Kerns and Székely [9]) stating that the law of X is a mixture of product probability measures with respect to a signed mixing measure. The result is "finitistic" in nature meaning that it is a matter of linear algebra for finite S. The passing from finite S to an arbitrary one may pose some measure-theoretic difficulties which are avoided by our proof. The mixing signed measure is not unique (examples are given), but we pay more attention to the one constructed in the proof ("canonical mixing measure") by pointing out some of its characteristics. The mixing measure is, in general, defined on the space of probability measures on S; but for $S = \mathbb{R}$, one can choose a mixing measure on $\mathbb{R}^n$.

1. Introduction

The first result that comes to mind when talking about exchangeability is de Finetti's theorem concerning sequences $X = (X_1, X_2, \ldots)$ of random variables with values in some space S and which are invariant under permutations of finitely many coordinates. This remarkable theorem [7, Theorem 11.10] states that the law of such a sequence is a mixture of product measures: let S^∞ be the product of countably many copies of S and let π^∞ be the product measure on S^∞ with marginals $\pi \in \mathscr{P}(S)$ (the space of probability measures on S); then

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$$\mathbb{P}(X \in \cdot) = \int_{\mathscr{P}(S)} \pi^\infty(\cdot)\, \nu(\mathrm{d}\pi),$$

for a uniquely defined probability measure ν which we call a mixing (or directing) measure.

In Bayesian language, this says that any exchangeable random sequence is obtained by first picking a probability distribution π from some prior (probability distribution on the space of probability distributions) and then letting the X_i to be i.i.d. with common law π. As Dubins and Freedman [5] show, de Finetti's theorem does not hold for an arbitrary measurable space S. Restrictions are required. One of the most general cases for which the theorem does hold is that of a Borel space S, i.e., a space which is isomorphic (in the sense of existence of a measurable bijection with measurable inverse) to a Borel subset of $\mathbb{R}$. Indeed, one of the most elegant proofs of the theorem can be found in Kallenberg [8, Section 1.1] from which it is evident that the main ingredient is the ergodic theorem and that the Borel space is responsible for the existence of regular conditional distributions.

For finite dimension n, however, things are different. Let S be a set together with a σ-algebra $\mathscr{S}$, and let $X_1, \dots, X_n$ be measurable functions from a measure space $(\Omega, \mathscr{F})$ into $(S, \mathscr{S})$. Under a probability measure $\mathbb{P}$ on $(\Omega, \mathscr{F})$, assume that $X = (X_1, \dots, X_n)$ is such that $\sigma X := (X_{\sigma(1)}, \dots, X_{\sigma(n)})$ has the same law as $(X_1, \dots, X_n)$ for any permutation σ of $\{1, \dots, n\}$, i.e., that $\mathbb{P}(\sigma X \in B) = \mathbb{P}(X \in B)$ for all $B \in \mathscr{S}^n$, where $\mathscr{S}^n$ is the product σ-algebra on S^n. In such a case, we say that X is n-exchangeable (or simply exchangeable).

Example 1. Simple examples show that a finitely exchangeable random vector may not be a mixture of product measures. For instance, take $S = \{1, \dots, n\}$, with $n \geq 2$, and let $X = (X_1, \dots, X_n)$ take values in S^n such that $\mathbb{P}(X = x) = 1/n!$ when $x = (x_1, \dots, x_n)$ is a permutation of $(1, \dots, n)$, and $\mathbb{P}(X = x) = 0$ otherwise. Clearly, X is n-exchangeable. Suppose that the law of X is a mixture of product measures. Since the space of probability measures $\mathscr{P}(S)$ can naturally be identified with the set $\Sigma_n := \{(p_1, \dots, p_n) \in \mathbb{R}^n : p_1, \dots, p_n \geq 0,\ p_1 + \cdots + p_n = 1\}$, the assumption that the law of X is a mixture of product measures is equivalent to the following: there is a random variable $p = (p_1, \dots, p_n)$ with values in Σ_n such that $\mathbb{P}(X = x) = \mathbb{E}[\mathbb{P}(X = x|p)]$, where $\mathbb{P}(X = x|p) = p_{x_1} \cdots p_{x_n}$ for all $x_1, \dots, x_n \in S$. But then, for all $i \in S$, $0 = \mathbb{P}(X_1 = \cdots = X_n = i) = \mathbb{E}[p_i^n]$, implying that $p_i = 0$, almost surely, for all $i \in S$, an obvious contradiction.

However, Jaynes [6] showed that (for the $|S| = 2$ case) there is mixing provided that signed measures are allowed; see equation (1) below. Kerns and Székely [9] observed that the Jaynes result can be generalized to *an arbitrary* measurable space S, but the proof in [9] requires some further explicit arguments. In addition, [9] uses a non-trivial algebraic result without a proof. Our purpose in this note is to give an alternative, shorter, and rigorous proof of the representation result (see Theorem 1 below) but also to briefly discuss some consequences and open problems (Theorem 2 and Section 4). An independent proof of an algebraic result needed in the proof of Theorem 1 is presented in Appendix A as Theorem 3. To the best of our knowledge, the proof is new and, possibly, of independent interest.

Theorem 1 *(Finite exchangeability representation theorem). Let $X_1, \dots, X_n$ be random variables on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with values in a measurable space $(S, \mathscr{S})$. Suppose that the law of $X = (X_1, \dots, X_n)$ is exchangeable. Then there is a signed measure ξ on $\mathscr{P}(S)$*

$$\mathbb{P}(X \in A) = \int_{\mathscr{P}(S)} \pi^n(A)\, \xi(\mathrm{d}\pi), \quad A \in \mathscr{S}^n, \tag{1}$$

where π^n is the product of n copies of $\pi \in \mathscr{P}(S)$.