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The improved isoperimetric inequality and the Wigner caustic of planar ovals

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ABSTRACT

The classical isoperimetric inequality in the Euclidean plane $\mathbb{R}^2$ states that for a simple closed curve M of the length L_M, enclosing a region of the area A_M, one gets

$$L_M^2 \geqslant 4\pi A_M.$$

In this paper we present the *improved isoperimetric inequality*, which states that if M is a closed regular simple convex curve, then

$$L_M^2 \geqslant 4\pi A_M + 8\pi \left|\widetilde{A}_{E_{\frac{1}{2}}(M)}\right|,$$

where $\widetilde{A}_{E_{\frac{1}{2}}(M)}$ is an oriented area of the Wigner caustic of M, and the equality holds if and only if M is a curve of constant width. Furthermore we also present a stability property of the improved isoperimetric inequality (near equality implies curve nearly of constant width). The Wigner caustic is an example of an affine λ-equidistant (for $\lambda = \dfrac{1}{2}$) and the improved isoperimetric inequality is a consequence of certain bounds of oriented areas of affine equidistants.

1. Introduction

The *classical isoperimetric inequality* in the Euclidean plane $\mathbb{R}^2$ states that:

Theorem 1.1 *(Isoperimetric inequality). Let M be a simple closed curve of the length L_M, enclosing a region of the area A_M, then*

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$$L_M^2 \geqslant 4\pi A_M, \tag{1.1}$$

and the equality (1.1) holds if and only if M is a circle.

This fact was already known in ancient Greece. The first mathematical proof was given in the nineteenth century by Steiner [25]. After that, there have been many new proofs, generalizations, and applications of this famous theorem, see for instance [3,12,17,19–21,23,25], and the literature therein. In 1902 Hurtwiz [19] and later Gao [12] showed the *reverse isoperimetric inequality*.

Theorem 1.2 *(Reverse isoperimetric inequality). Let K be a strictly convex domain whose support function p has the property that p'' exists and is absolutely continuous, and let $\widetilde{A}$ denote the oriented area of the evolute of the boundary curve of K. Let L_K be the perimeter of K and A_K be the area of K. Then*

$$L_M^2 \leqslant 4\pi A_M + \pi|\widetilde{A}|. \tag{1.2}$$

Equality holds if and only if $p(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta$.

In this paper we present bounds of oriented areas of affine equidistants and thanks to it we will prove the *improved isoperimetric inequality*, which states that if M is a closed regular simple convex curve, then

$$L_M^2 \geqslant 4\pi A_M + 8\pi\left|\widetilde{A}_{E_{\frac{1}{2}}(M)}\right|,$$

where $\widetilde{A}_{E_{\frac{1}{2}}(M)}$ is an oriented area of the Wigner caustic of M, and the equality holds if and only if M is a curve of constant width. It is very interesting that the absolute value of the oriented area of the Wigner caustic improves the classical isoperimetric inequality and also gives the exact link between the area and the length of constant width curves.

The family of affine λ-equidistants arises as the counterpart of parallels or offsets in Euclidean geometry. An affine equidistant for us is the set of points of chords connecting points on M where tangent lines to M are parallel, which divide the chord segments between the base points with a fixed ratio λ, also called the *affine time*. When in affine λ-equidistants the ratio λ is equal to $\frac{1}{2}$ then this set is also known as the *Wigner caustic*. The Wigner caustic of a smooth convex closed curve on affine symplectic plane was first introduced by Berry, in his celebrated 1977 paper [2] on the semiclassical limit of Wigner's phase-space representation of quantum states. There are many papers considering affine equidistants, see for instance [4–9,13,14,22,26], and the literature therein. The Wigner caustic is also known as the *area evolute*, see [4,13].

2. Geometric quantities, affine equidistants and Fourier series

Let M be a smooth planar curve, i.e. the image of the C^∞ smooth map from an interval to $\mathbb{R}^2$. A smooth curve is *closed* if it is the image of a C^∞ smooth map from S^1 to $\mathbb{R}^2$. A smooth curve is *regular* if its velocity does not vanish. A regular closed curve is *convex* if its signed curvature has a constant sign. An *oval* is a smooth closed convex curve which is simple, i.e. it has no selfintersections. In our case it is enough to consider C^2-smooth curves.

Definition 2.1. A pair $a, b \in M$ $(a \neq b)$ is called the *parallel pair* if tangent lines to M at points a, b are parallel.

Definition 2.2. A *chord* passing through a pair $a, b \in M$ is the line:

$$l(a,b) = \left\{\lambda a + (1-\lambda)b \,\middle|\, \lambda \in \mathbb{R}\right\}.$$