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Four-dimensional CR submanifolds of the sphere $\mathbf{S}^6(1)$ with two-dimensional nullity distribution

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ABSTRACT

We investigate four-dimensional CR submanifolds of the nearly Kähler sphere $\mathbf{S}^6(1)$ with nullity distribution of the maximal possible dimension two, and classify them using a sphere curve and a vector field along that curve.

1. Introduction

It is well known that from the multiplication of the Cayley numbers $\mathcal{O}$, there arises a cross product $\times$ in $\mathbb{R}^7 = Im\mathcal{O}$ and an almost complex structure on the standard unit sphere $\mathbf{S}^6(1) \subset \mathbb{R}^7$ which makes it a nearly Kähler manifold. Its group of isometries is the exceptional Lie group G_2.

It is natural to study submanifolds of the manifold with almost complex structure, with respect to that structure. If the tangent space of the submanifold is invariant for J, it is called an almost complex submanifold. If the tangent space is by J mapped into the corresponding normal space, it is called a totally real submanifold. A generalization of this is the notion of CR submanifold as introduced by A. Bejancu in [4].

A submanifold M of $\mathbf{S}^6(1)$ is called a CR submanifold if there exists a C^∞-differential almost complex distribution $U : x \to U_x \subset T_xM$, i.e. $JU \subset U$ on M, such that its orthogonal complement $U^\perp$ in TM is a totally real distribution, i.e. $JU^\perp \subset T^\perp M$, where $T^\perp M$ is the normal bundle over M in $\mathbf{S}^6(1)$. We say that M is proper if neither the almost complex, nor the totally real distribution are trivial. CR submanifolds have been previously studied amongst others by K. Mashimo, H. Hashimoto and K. Sekigawa (see [9] and [8]).

The four-dimensional CR submanifolds of $\mathbf{S}^6(1)$ can not be totally geodesic. Therefore, it is natural to investigate submanifolds with nullity distribution

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$$\mathcal{D}(p) = \{X \in T_pM | h(X,Y) = 0, \forall Y \in T_pM\}, \tag{1}$$

of the maximal possible dimension. In [5] the three-dimensional CR submanifolds of the sphere $\mathbf{S}^6(1)$ with nullity distribution of the maximal dimension, which is one, were classified by constructions that start from one or two curves in $\mathbf{S}^6(1)$. If the second fundamental form vanishes on a distribution, then it is called a totally geodesic distribution. In [3] a class of four-dimensional CR submanifolds that locally admit a particular kind of twisted product structure was investigated. It was shown that if $A(t)$ is a curve in the Lie group G_2 and $f(u,v,w)$ a three-dimensional totally real submanifold of $\mathbf{S}^6(1)$, then the map $A(t)f(u,v,w)$, provided that it is an immersion, is a CR immersion. In particular, a four-dimensional CR submanifold of $\mathbf{S}^6(1)$ having its totally real distribution totally geodesic and with a two-dimensional nullity distribution is locally congruent to the immersion

$$F_1(y_1,y_2,y_3,y_4,s) = A(s)(y_1,0,y_2,0,y_3,0,y_4), \tag{2}$$

where $y_1^2+y_2^2+y_3^2+y_4^2=1$, and the G_2 curve $A(s)$ is given by

$$A(s) = \big(\gamma, \quad A_3\times\gamma, \quad A_3, \quad (A_3\times\gamma'),\times\gamma, \quad A_3\times\gamma', \quad \gamma', \quad -\gamma\times\gamma'\big)(s)$$

where γ is a unit length sphere curve satisfying

$$A_3'\times A_3 \perp \gamma, \gamma\times\gamma', \quad A_3 - \langle A_3', \gamma'\rangle\gamma \perp \gamma''\times\gamma'. \tag{3}$$

Here, we prove the following theorem.

Theorem 1. *Let M be a four-dimensional CR submanifold of the sphere $\mathbf{S}^6(1)$ with a two-dimensional nullity distribution. Then it is locally congruent to the immersion (2) with conditions (3), or to the immersion*

$$\begin{aligned}F(x_1,x_2,x_3,s) = A(s)(&\sin x_2, \sin x_1\cos x_2, 0, \cos x_1\cos x_2\cos f_1, 0,\\ &\frac{2}{\sqrt{4+m^2}}\cos x_1\cos x_2\sin f_1, -\frac{m}{\sqrt{4+m^2}}\cos x_1\cos x_2\sin f_1),\end{aligned}$$

where $\cos x_1, \cos x_2 > 0$, m is a constant, f_1 is a function of x_3 and s such that $\partial_{x_3}f_1 > 0$, and $A(s)$ is a G_2-curve given by

$$A(s) = \big(L, \quad L', \quad L\times L', \quad B_1, \quad L\times B_1, \quad L'\times B_1, \quad -(L\times B_1)\times L'\big)(s),$$

where L is a sphere curve parameterized by its arc length s, such that $\langle L'', L\times L'\rangle = 0$, B_1 is a unit vector field along L, orthogonal to $L''\times L$ such that $\langle B_1'\times B_1, L\rangle = 0$ and $\langle L''\times L', mB_1 + 2L\times B_1\rangle = 0$.

2. Preliminaries

The multiplication of the Cayley numbers $\mathcal{O} = \mathbb{R}^8$ can be used to define a vector cross product $\times$ on the set of the purely imaginary Cayley numbers $\mathbb{R}^7$ in the following way

$$u\times v = \frac{1}{2}(uv - vu).$$

This cross product has many similarities with the cross product in $\mathbb{R}^3$, for instance, the triple scalar product $\langle u\times v, w\rangle$ is skew symmetric in u, v, w where $\langle , \rangle$ denotes the standard inner product in the space $\mathbb{R}^7$. Also, see [7], we have that