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Geometric approach for global asymptotic stability for three species competitive Gompertz models ☆

Guichen Lu [a,*], Zhengyi Lu [b]

[a] School of Mathematics and Statistics, Chongqing University of Technology, Chongqing 400054, China
[b] Department of Mathematics, Sichuan Normal University, Chengdu 610066, China

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ABSTRACT

This paper deals with the three dimensional competitive models with Gompertz growth in population dynamics. By using the geometric criterion and time average property to Gompertz models, a new criterion on the global asymptotical stability of the models is established. Furthermore, based on this criterion, a partial affirmative answer to Jiang–Niu–Zhu's open problems is given.

1. Introduction

Among various approaches to describe the rate of population growth, Logistic Law developed by Verhulst in 1838 was widely used in population dynamics. For example, based on the Logistic law, Vito Volterra in 1925 proposed the classical Lotka–Volterra model as an explanation of data from some fish studies of D'Ancona. The same model was independently produced by Alfred Lotka in 1920. After noticing Volterra's work, Kolmogorov improved more general predator and prey population model in 1936 which is often called Kolmogorov's predator–prey model [8]. However, these results cannot suggest the logistic growth as good fit for any particular class of "real-word" population dynamics, even those which seem to be isolated. For instance, to describe the human mortality [9,19] and the growth of tumor [14,23], it is appropriate to use Gompterz growth equation rather than using the logistic growth equation.

We shall give a derivation for the Gompertz model here. Let's denote by $x(t)$ the size of the tumor population. The Gompertz growth law can be described by the following coupled equations:

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* Corresponding author.
E-mail address: guichenlu@cqut.edu.cn (G. Lu).

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$$\begin{aligned} \dot{x}(t) &= x(t)r(t), \\ \dot{r}(t) &= -ar(t), \end{aligned} \tag{1.1}$$

with the initial conditions $x(0) = x_0$ and $r(0) = r_0$, where $r(t)$ is the tumor growth rate and $a > 0$ is a retardation constant.

The solution is given by

$$x(t) = x_0 e^{(\frac{r_0}{a}(1-e^{-at}))}, \tag{1.2}$$

where the constants can be determined by the data. When considering the clinic treatment, we assume that the post-treatment tumor has the same growth characteristics as the original tumor. For simplify, we assume that the tumor remains unchanged in size, we denote this as $b = x_0 \exp(\frac{r_0}{a})$. Equations (1.1) can be written in the following single form:

$$\dot{x}(t) = ax(t)\ln(\frac{b}{x(t)}). \tag{1.3}$$

This was originally developed by Gompertz [9]. Nowadays, Gompertz growth model is widely used by ecologists to explain biological phenomena [2,3,5,13,17,22–24].

When the competition is among several species, Yu, Wang and Lu [25] proposed the following three-dimensional Gompertz model

$$\begin{aligned} \dot{x}_1 &= x_1[\ln b_1 - \ln(x_1 + a_{12}x_2 + a_{13}x_3)], \\ \dot{x}_2 &= x_2[\ln b_2 - \ln(a_{21}x_1 + x_2 + a_{23}x_3)], \\ \dot{x}_3 &= x_3[\ln b_3 - \ln(a_{31}x_1 + a_{32}x_2 + x_3)]. \end{aligned} \tag{1.4}$$

Here x_i $(i = 1, 2, 3)$ are the population densities of the i-th species at time t. b_i represents the carrying capacity of i-th population; a_{ii} $(i = 1, 2, 3)$ are the rates of intra-specific competition of the first, second and third species, respectively; a_{ij} $(i \neq j)$ are the rates of inter-specific competition of the population j on the population i.

To obtain the dimensionless form of the model (1.4), we take the following:

$$x_i' = \frac{x_i}{b_i},\ b_{12} = \frac{b_{12}b_1}{b_2},\ b_{13} = \frac{a_{13}b_3}{b_1},\ b_{21} = \frac{a_{21}b_1}{b_2},\ b_{23} = \frac{a_{23}b_3}{b_2},\ b_{31} = \frac{a_{31}b_1}{b_3},\ b_{32} = \frac{a_{32}b_2}{b_3}.$$

Dropping the primes, model (1.4) becomes

$$\begin{aligned} \dot{x}_1 &= x_1 \ln \frac{1}{x_1 + b_{12}x_2 + b_{13}x_3}, \\ \dot{x}_2 &= x_2 \ln \frac{1}{b_{21}x_1 + x_2 + b_{23}x_3}, \\ \dot{x}_3 &= x_3 \ln \frac{1}{b_{31}x_1 + b_{32}x_2 + x_3}. \end{aligned} \tag{1.5}$$

Due to the biological interpretation, we assume that model (1.5) has an unique positive equilibrium $x^* = (x_1^*, x_2^*, x_3^*)$ with

$$x_1^* = \frac{\Delta_1}{\Delta},\ x_2^* = \frac{\Delta_2}{\Delta},\ x_3^* = \frac{\Delta_3}{\Delta},$$