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Truncation and spectral variation in Banach algebras

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Let a and b be elements of a semisimple, complex and unital Banach algebra A. Using subharmonic methods, we show that if the spectral containment $\sigma(ax) \subseteq \sigma(bx)$ holds for all $x \in A$, then ax belongs to the bicommutant of bx for all $x \in A$. Given the aforementioned spectral containment, the strong commutation property then allows one to derive, for a variety of scenarios, a precise connection between a and b. The current paper gives another perspective on the implications of the above spectral containment which was also studied, not long ago, by J. Alaminos, M. Brešar et al.

1. Introduction

Problems related to spectral variation under the multiplicative and additive operations in Banach algebras have recently attracted attention of researchers working in the field of abstract spectral theory in Banach algebras. Specifically, the first contributions were made by Brešar and Špenko [7], and at around the same time, but independently by Braatvedt and Brits [5], and then later by J. Alaminos et al. [1], and Brits and Schulz [9]. The aim of this paper is to extend and elaborate on the results obtained in [1] and [9]; we shall employ techniques which are distinctly different from the methods used in [1] and [7].

Unless otherwise stated, A will assumed to be a semisimple, complex, and unital Banach algebra with the unit denoted by $\mathbf{1}$. The group of invertible elements, and the centre of A are denoted respectively by $G(A)$ and $Z(A)$. We shall use σ_A and ρ_A to denote, respectively, the spectrum

$$\sigma_A(x) := \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin G(A)\},$$

and the spectral radius

$$\rho_A(x) := \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$$

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of an element $x \in A$ (and agree to omit the subscript if the underlying algebra is clear from the context). Denote further by $\sigma'(x) := \sigma(x)\backslash\{0\}$ the *non-zero spectrum* of $x \in A$. If X is a compact Hausdorff space, then $A = C(X)$ is the Banach algebra of continuous, complex functions on X with the usual pointwise operations and the spectral radius as the norm. If X is a complex Banach space then $A = \mathcal{L}(X)$ is the Banach algebra of bounded linear operators on X to X (also in the usual sense). The main question of this paper is, loosely stated, the following:

Let A be a semisimple, complex, and unital Banach algebra, and suppose that $a, b \in A$ satisfy

$$\sigma(ax) \subseteq \sigma(bx) \text{ for all } x \in A. \tag{1.1}$$

What is the relationship between a and b?

Observe, trivially, that

$$(1.1) \Rightarrow \sigma'(ax) \subseteq \sigma'(bx) \text{ for all } x \in A.$$

Since the non-zero spectrum is cyclic (Jacobson's Lemma, [3, Lemma 3.1.2]) it turns out to be advantageous to assume, where applicable, the preceding implication of (1.1) rather than (1.1) itself. For easy reference we label

$$\sigma'(ax) \subseteq \sigma'(bx) \text{ for all } x \in A, \tag{1.2}$$

and then note that (1.2) is equivalent to the statement:

$$\sigma'(xa) \subseteq \sigma'(xb) \text{ for all } x \in A.$$

Further, if (1.2) holds then we also have

$$\sigma'((b-a)x) \subseteq \sigma'(bx) \text{ for all } x \in A. \tag{1.3}$$

To see this, if $\lambda \neq 0$ and $\lambda \notin \sigma'(bx)$, then

$$1 + bx(\lambda\mathbf{1} - bx)^{-1} = \lambda(\lambda\mathbf{1} - bx)^{-1} \in G(A),$$

from which the assumption (1.2) implies that $\mathbf{1} + ax(\lambda\mathbf{1} - bx)^{-1} \in G(A)$. Then

$$\lambda\mathbf{1} - (b-a)x = (\mathbf{1} + ax(\lambda\mathbf{1} - bx)^{-1})(\lambda\mathbf{1} - bx) \in G(A).$$

We give a short list of some of the major known results which are related to (1.1) and (1.2):

(a) [7, Theorem 3.7]: Let A be a prime $C^\star$-algebra and let $a, b \in A$ be such that $\rho(ax) \leq \rho(bx)$ for all $x \in A$. Then there exists $\lambda \in \mathbb{C}$ such that $|\lambda| \leq 1$ and $a = \lambda b$.

(b) [5, Theorem 2.6]: If A is an arbitrary semisimple, complex and unital Banach algebra, and $a, b \in A$, then $a = b$ if and only if $\sigma(ax) = \sigma(bx)$ for all $x \in A$ satisfying $\rho(x - \mathbf{1}) < 1$ (the bound on the spectral radius is sharp).

(c) [1, Theorem 2.3]: If A is a unital $C^\star$-algebra and $a, b \in A$, then $\sigma(ax) \subseteq \sigma(bx) \cup \{0\}$ for every $x \in A$ if and only if there exists a central projection $z \in A''$, the second dual of A, such that $a = zb$.

(d) [1, Theorem 3.6]: If A is a unital $C^\star$-algebra and $a, b \in A$, then $\rho(ax) \leq \rho(bx)$ for every $x \in A$ if and only if there exists a central projection $z \in A''$, the second dual of A, such that $a = zb$ and $\|z\| \leq 1$.