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Operator Positivstellensätze for noncommutative polynomials positive on matrix convex sets

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ABSTRACT

This article studies algebraic certificates of positivity for noncommutative (nc) operator-valued polynomials on matrix convex sets, such as the solution set D_L, called a free Hilbert spectrahedron, of the linear operator inequality (LOI) $L(X) = A_0 \otimes I + \sum_{j=1}^{g} A_j \otimes X_j \succeq 0$, where A_j are self-adjoint linear operators on a separable Hilbert space, X_j matrices and I is an identity matrix. If A_j are matrices, then $L(X) \succeq 0$ is called a linear matrix inequality (LMI) and D_L a free spectrahedron. For monic LMIs, i.e., $A_0 = I$, and nc matrix-valued polynomials the certificates of positivity were established by Helton, Klep and McCullough in a series of articles with the use of the theory of complete positivity from operator algebras and classical separation arguments from real algebraic geometry. Since the full strength of the theory of complete positivity is not restricted to finite dimensions, but works well also in the infinite-dimensional setting, we use it to tackle our problems. First we extend the characterization of the inclusion $D_{L_1} \subseteq D_{L_2}$ from monic *LMIs* to monic *LOIs* L_1 and L_2. As a corollary one immediately obtains the description of a polar dual of a free Hilbert spectrahedron D_L and its projection, called a free Hilbert spectrahedrop. Further on, using this characterization in a separation argument, we obtain a certificate for multivariate matrix-valued nc polynomials F positive semidefinite on a free Hilbert spectrahedron defined by a monic LOI. Replacing the separation argument by an operator Fejér–Riesz theorem enables us to extend this certificate, in the univariate case, to operator-valued polynomials F. Finally, focusing on the algebraic description of the equality $D_{L_1} = D_{L_2}$, we remove the assumption of boundedness from the description in the LMIs case by an extended analysis. However, the description does not extend to LOIs case by counterexamples.

1. Introduction

In this section we state the main concepts and results of this paper. Subsection 1.1 places the content of the paper in a general context. In Subsections 1.2–1.6 definitions intertwine with the main results. Subsection 1.7 is a guide to the organization of the rest of the paper.

Throughout the paper $\mathscr{H}$, $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{K}$, $\mathscr{G}$ stand for separable real Hilbert spaces unless stated otherwise.

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1.1. Context

The name Positivstellensatz refers to an algebraic certificate for a given polynomial p to have a positivity property on a given closed semialgebraic set. Finding a certificate for an operator-valued polynomial p positive semidefinite on an arbitrary closed semialgebraic set is a hard problem. Even if p is a matrix-valued polynomial, the optimal certificates are only known to exist for very special sets, namely matrix convex sets defined as matrix solution sets of linear matrix inequalities (LMIs). The aim of this paper is to generalize characterizations of noncommutative (nc) matrix-valued polynomials which are positive semidefinite on a LMI set to characterizations of nc operator-valued polynomials which are positive semidefinite on arbitrary matrix convex sets. By [21], every closed matrix convex set is a matrix solution set of a linear operator inequality (LOI).

Our problem belongs to the field of free real algebraic geometry (free RAG); see [30] and references therein. Free RAG has two branches – free positivity and free convexity. Both branches present exciting mathematical challenges, and lend themselves to many applications.

Free positivity is an analog of classical real algebraic geometry [10,39–41,48,49,51], a theory of polynomial inequalities embodied in Positivstellensätze. It makes contact with noncommutative real algebraic geometry [12,25,31,29,26,42,52]. Free Positivstellensätze have applications to quantum physics [46], operator algebras [36], quantum statistical mechanics [37,11], the quantum moment problems and multiprover games [19].

Matrix convex sets and free convexity arise naturally in a number of contexts, including engineering systems theory, operator spaces, systems and algebras and is closely linked to unital completely positive maps [4,57,56,22,45,28,23]. The simplest examples of matrix convex sets are matrix solution sets of LMIs. A large class of linear systems engineering problems transforms to LMIs [30, §1.1], which led to a major advance in those problems during the past two decades [53]. Furthermore, LMIs underlie the theory of semidefinite programming, an important recent innovation in convex optimization [44]. As mentioned above every closed matrix convex set is a matrix solution sets of a LOI by [21].

1.2. Free sets, matrix convex sets, linear pencils and LOI sets

This work fits into the wider context of free analysis [54,55,35,43,47,1,7,17,31,46], so we start by recalling some of the standard notions used throughout this article.

1.2.1. Free sets – matrix level

Fix a positive integer $g \in \mathbb{N}$. We use $\mathbb{S}_n$ to denote real symmetric $n \times n$ matrices and $\mathbb{S}^g$ for the sequence $(\mathbb{S}_n^g)_n$. A **subset** Γ of $\mathbb{S}^g$ is a sequence $\Gamma = (\Gamma(n))_n$, where $\Gamma(n) \subseteq \mathbb{S}_n^g$ for each n. The subset Γ is **closed with respect to direct sums** if $A = (A_1, \dots, A_g) \in \Gamma(n)$ and $B = (B_1, \dots, B_g) \in \Gamma(m)$ implies

$$A \oplus B = \left(\begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix}, \dots, \begin{bmatrix} A_g & 0 \\ 0 & B_g \end{bmatrix} \right) \in \Gamma(n+m).$$

It is closed with respect to **(simultaneous) unitary conjugation** if for each n, each $A \in \Gamma(n)$ and each $n \times n$ unitary matrix U,

$$U^* A U = (U^* A_1 U, \dots, U^* A_g U) \in \Gamma(n).$$

The set Γ is a **free set** if it is closed with respect to direct sums and simultaneous unitary conjugation. If in addition it is closed with respect to **(simultaneous) isometric conjugation**, i.e., if for each $m \le n$, each $A = (A_1, \dots, A_g) \in \Gamma(n)$, and each isometry $V : \mathbb{R}^m \to \mathbb{R}^n$,