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On the local well-posedness for the nonlinear Schrödinger equation with spatial variable coefficient

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ABSTRACT

We study a new type of nonlinear Schrödinger equation where the coefficient of Laplacian depends on spatial variable. Based on a modified Hankel transform and delicate frequency estimates, we establish the local well-posedness of the NLS with spatial variable coefficient in the weighted Lebesgue space for $n \geq 2$, and extend the Strichartz estimates to the non-radially Schrödinger equation with spatial variable coefficient in 2D Euclidian space.

1. Introduction and motivation

We consider the continuum model of n-spatial inhomogeneous Heisenberg ferromagnetic spin system (HFSS) with nearest-neighbor interaction, which can be written as

$$\vec{S}_t(\mathbf{x}, t) = \varrho(\mathbf{x})\vec{S} \times \nabla^2\vec{S} + \nabla\varrho(\mathbf{x}) \cdot (\vec{S} \times \nabla\vec{S}). \tag{1.1}$$

Here $\varrho(\mathbf{x})$ is a scalar function, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is an n-dimension vector, and $\nabla^2 = \nabla \cdot \nabla$ is n-spatial Laplacian, while the spin $\vec{S} = (S^x, S^y, S^z)$ is constrained by $\vec{S}^2 = 1$. When $\varrho \equiv 1$, it is known as the Schrödinger map (see [13,20–22] and references there for detail).

Choosing to study the dynamics of the spherically symmetric version, the corresponding equation of motion is

$$\vec{S}_t(r, t) = \varrho(r)\vec{S} \times [\vec{S}_{rr} + \frac{n-1}{r}\vec{S}_r] + \varrho_r(r)[\vec{S} \times \vec{S}_r], \tag{1.2}$$

where $r = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}},\ 0 < r < \infty$.

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Mapping the spin vector $\vec{S}$ onto a moving helical space curve in $\mathbb{R}^3$ with curvature and torsion

$$\kappa(r,t) = [\vec{S}_r \cdot \vec{S}_r]^{\frac{1}{2}}, \qquad \tau = \kappa^{-2}\vec{S} \cdot (\vec{S}_r \times \vec{S}_{rr}), \tag{1.3}$$

the time evolution equations for the orthogonal trihedral are given by

$$\vec{e}_{jr} = (\tau\vec{e}_1 + \kappa\vec{e}_3) \times \vec{e}_j, \qquad \vec{e}_{jt} = \vec{e}_j \times (\omega_1\vec{e}_1 + \omega_2\vec{e}_2 + \omega_3\vec{e}_3), \qquad j = 1,2,3, \tag{1.4}$$

where

$$\begin{aligned}
\omega_1 &= \frac{1}{\kappa}[\varrho\kappa_{rr} + (\frac{n-1}{r}\varrho + 2\varrho_r)\kappa_r - \varrho\kappa\tau^2 + (\varrho_{rr} + \frac{n-1}{r}\varrho_r - \frac{n-1}{r^2}\varrho)\kappa],\\
\omega_2 &= -[\varrho\kappa_r + (\varrho_r + \frac{n-1}{r}\varrho)\kappa],\\
\omega_3 &= -\varrho\kappa\tau.
\end{aligned}$$

Through a complex transformation

$$v(r,t) = \frac{\kappa}{2}\exp\{i\int_0^r \tau(r',t)dr'\},$$

it leads to the following generalized nonlinear Schrödinger equations (GNLS):

$$\begin{aligned}
&iv_t + \varrho(v_{rr} + \frac{n-1}{r}v_r - \frac{n-1}{r^2}v + 2|v|^2 v)\\
&\quad + 2\varrho_r v_r + [\varrho_{rr} + \frac{n-1}{r}\varrho_r + 2\int_0^r \varrho_{r'}|v|^2 dr' + 4(n-1)\int_0^r \frac{\varrho}{r'}|v|^2 dr']v = 0.
\end{aligned} \tag{1.5}$$

From the above geometrical approach, we obtain the spin evolution Equation (1.2) is equivalent to the GNLS (1.5). Through the Painlevé singularity structure analysis ([7]), the spherically symmetric spin system is integrable when the inhomogeneity

$$\varrho(r) = \epsilon_1 r^{-2(n-1)} + \epsilon_2 r^{-(n-2)}, \tag{1.6}$$

where ϵ_1, ϵ_2 are arbitrary constants. In particular, when $\epsilon_1 = 0, n = 2$ and $\epsilon_2 = 1$, the GNLS (1.5) becomes classical NLS as

$$\begin{cases} i\partial_t u(\mathbf{x},t) - \Delta u(\mathbf{x},t) = F(u),\\ u(\mathbf{x},0) = u_0(\mathbf{x}). \end{cases} \tag{1.7}$$

Motivated by the integrable model (1.5), as the first step, we try to exploit the well-posedness of the solution to the initial value problem (IVP) for the derived NLS equation with spatial variable coefficient:

$$\begin{cases} iv_t - r^{p_0}(v_{rr} + \frac{p_1}{r}v_r - \frac{p_2}{r^2}v) = f(r,v),\\ v(r,0) = v_0(r), \end{cases} \tag{1.8}$$

where v is a radial complex function, $r = |\mathbf{x}|(\mathbf{x} \in \mathbb{R}^n)$ is the radial radius and f is a nonlinear complex valued function, the index p_m $(m = 0,1,2)$ are independent of the variables r, t.

This paper is organized as follows: In Section 2, we mention the major results. In Section 3, we present the integral representation and crucial properties of the Hankel transform, which is the main tool for proving the