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Existence and nonexistence of global solutions for a semilinear reaction–diffusion system

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ABSTRACT

This paper is concerned with the blow-up and global existence of nonnegative solutions to the following Cauchy problem

$$\begin{aligned} u_t - \Delta u &= v^p, \qquad t > 0, x \in R^N, \\ v_t - \Delta v &= a(x)u^q, \qquad t > 0, x \in R^N, \\ u(x,0) = u_0(x), \quad v(x,0) &= v_0(x), \qquad x \in R^N, \end{aligned}$$

where the constants $p, q > 0$ and $a(x) \gneqq 0$ is on the order $|x|^m$ as $|x| \to \infty$, $m \in \mathbb{R}$. The Fujita critical exponent is determined when $m \geq 0$, and some results of global existence of solution under some assumptions when $m < 0$ are also obtained. The results extend those in Escobedo and Herrero (1991) [9] and indicate that m affects the Fujita critical exponent.

1. Introduction

Since the pioneering work of Fujita [14,15], a large amount of work has been devoted to studying the critical exponents, the conditions which guarantee the global existence or the blow-up of solutions for partial differential equations, one can refer to [7,11,12,20,26,33,34] for semilinear parabolic equations, [4,27,28,40] for fast diffusion equations, [19,39] for porous medium equations, [6,9,10,29] for parabolic systems and [22, 30] for other equations. Meanwhile, other relevant problems, such as blow-up rate, have also been paid much attention, refer to [2,5,13,16,21,24,36–38] and references therein. In recent years, there are also many works

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devoted to the so-called second critical exponent which describes the critical sizes of initial data required by global (non-global) solution via the decay order of the initial data at space-infinity in the coexistence region of global and non-global solutions, determined by the critical Fujita exponents [1,17,25,31,32,35,41].

This paper is concerned with the existence and nonexistence of nonnegative global solutions for the semilinear reaction–diffusion system

$$\begin{cases} u_t - \Delta u = v^p, & t > 0, x \in \mathbb{R}^N, \\ v_t - \Delta v = a(x)u^q, & t > 0, x \in \mathbb{R}^N, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geqslant 1,\ p, q > 0$ and $0 \lneqq a(x) \in C^\alpha(\mathbb{R}^N)$ is on the order $|x|^m$, when $|x|$ is large enough, $m \in (-\infty, \infty)$. $u_0(x)$ and $v_0(x)$ are nonnegative, continuous and bounded functions in $\mathbb{R}^n$.

The existence of solutions $(u(x,t), v(x,t))$ of (1.1) with small time T can be guaranteed by a standard fixed point argument. Moreover, since the nonlinearity, v^p is locally Lipschitz in v and $a(x)u^q$ is locally Hölder continuous in x and locally Lipschitz in u, so by the classical parabolic equation results, these solutions are actually classical, that is, $u(x,t), v(x,t) \in C^{2,1}(R^N \times (0,T))$. We omit the detailed proof and one can refer to similar results in [8,9].

As we know, for the problem

$$\begin{cases} u_t - \Delta u = a(x)u^p, & t > 0, x \in \mathbb{R}^N, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

when $a(x) \equiv 1$, Fujita [14,15] proved that if $p > 1+\frac{2}{N}$, (1.2) can possess both global and nonglobal solutions, and if $1 < p < 1 + \frac{2}{N}$, (1.2) can not possess any global solution. For the critical case $p = 1 + \frac{2}{N}$ which belongs to the nonexistence of global solutions, it was showed in [18] for $N = 1, 2$ and in [3,23] for $N \geqslant 1$. The critical exponent $1 + \frac{2}{N}$ is called the Fujita exponent which can separate when the problem exists or does not exist global solutions.

For the general $a(x)$, in [33], Pinsky found that $a(x)$ would effect the Fujita exponent by changing it from $1 + \frac{2}{N}$ to $1 + \frac{2+m}{N}$.

For the problem (1.1), in the special case of $a(x) \equiv 1$, Escobedo and Herrero [9] delivered the results that this problem also has a Fujita exponent to separate the existence and nonexistence of global solutions. They obtained that if $0 < pq \leqslant 1$, every solution of (1.1) is global; if $pq > 1$ and $(\max\{p,q\}+1)/(pq-1) \geqslant N/2$, every nontrivial solution of (1.1) blows up in a finite time; if $pq > 1$ and $(\max\{p,q\}+1)/(pq-1) < N/2$, then (1.1) possesses both global and nonglobal solutions.

Motivated by the above results, in this paper, we consider the nonnegative classical solutions $(u(x,t), v(x,t))$ of (1.1) in some strip $S_T = [0,T) \times R^N$ and our goal is to study how $a(x)$ effects the Fujita exponent. For short, $(u(x,t), v(x,t))$ will be often denoted in abridged way $(u(t), v(t))$.

Compared with the scalar problem (1.2), system (1.1) seems more complicated. Such as due to the coupling of variables, we can not use the method that treating the single equations to get the upper estimates of the solutions of (1.1), thus we use a more complicate iteration method to get a proper upper estimates. Moreover, our results include the results of [9] where $a(x) \equiv 1$ and find that the inhomogeneous term $a(x)$ will affect the Fujita critical exponent.

Define

$$T^* \equiv T^*(u,v) = \sup\{T > 0 : (u(t), v(t)) \text{ is bounded in } S_T \text{ and satisfies (1.1) }\}. \tag{1.3}$$