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On the Cauchy problem for a class of shallow water wave equations with $(k+1)$-order nonlinearities ☆

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A B S T R A C T

This paper considers the Cauchy problem for a class of shallow water wave equations with $(k+1)$-order nonlinearities in the Besov spaces

$$\partial_t u - \partial_t \partial_x^2 u = u^k \partial_x^3 u + b u^{k-1} \partial_x u \partial_x^2 u - (b+1) u^k \partial_x u,$$

which involves the Camassa–Holm, the Degasperis–Procesi and the Novikov equations as special cases. Firstly, by means of the transport equation and the Littlewood–Paley theory, we obtain the local well-posedness of the equations in the nonhomogeneous Besov space $B^s_{p,r}$ ($s > \max\{1+\frac{1}{p}, \frac{3}{2}\}$ and $p, r \in [1, +\infty]$). Secondly, we consider the local well-posedness in $B^s_{2,r}$ with the critical index $s = \frac{3}{2}$, and show that the solutions continuously depend on the initial data. Thirdly, the blow-up criteria and the conservative property for the strong solutions are derived. Finally, with the help of a new Ovsyannikov theorem, we investigate the Gevrey regularity and analyticity of the solutions. Moreover, we get a lower bound of the lifespan and the continuity of the data-to-solution mapping.

1. Introduction

In this paper, we consider the Cauchy problem for a class of shallow water wave equations with $(k+1)$-order nonlinearities,

$$\partial_t u - \partial_t \partial_x^2 u = u^k \partial_x^3 u + b u^{k-1} \partial_x u \partial_x^2 u - (b+1) u^k \partial_x u, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

which was recently proposed by Himonas and Holliman in [29]. Here, k is a given positive integer number, and the parameter b is assumed to range over the real line $\mathbb{R}$. As special cases, the Camassa–Holm, the

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Degasperis–Procesi and the Novikov equations are integrable members of this family of equations. By means of a Galerkin type approximation method, the local well-posedness of (1.1) in the Sobolev spaces $C([0,T];H^s(\mathbb{R}))\cap C^1([0,T];H^{s-1}(\mathbb{R}))$ is established [29], and the data-to-solution mapping is proved to be continuous but not uniformly continuous on any bounded subset of $H^s(\mathbb{R})$ $(s>\frac{3}{2})$. Furthermore, Holmes [32] proved that the data-to-solution mapping is Hölder continuous in $H^s(\mathbb{R})$ $(s>\frac{3}{2})$ endowed with the H^r-topology for $0\le r<s$, and the Hölder exponent is expressed in terms of s and r. On the other hand, if we write the Eq. (1.1) in the nonlocal form ((1.7) below), it can be regarded as a weakly dispersive perturbation of the generalized Burgers equation $\partial_t u+u^k\partial_x u=0$. One of the important properties that makes the Eq. (1.1) an interesting evolution equation is that the Burgers equation has no peakon traveling wave solutions (called peakons) while the Eq. (1.1) does. More precisely, on the line $\mathbb{R}$, the peakons are given by

$$u_c(x,t)=c^{\frac{1}{k}}e^{|x-ct|}, \tag{1.2}$$

and on the circle $\mathbb{S}$, the peakons are given by

$$u_c(x,t)=\frac{c^{\frac{1}{k}}}{\cosh(\pi)}\cosh([x-ct]_p-\pi), \tag{1.3}$$

where c is any positive constant and $[x-ct]_p:=x-ct-2\pi[\frac{x-ct}{2\pi}]$ in [27]. By using these peakon solutions, one can construct two sequences of solutions whose distance at the initial time goes to zero while at any later time their distance goes to infinitely in the Sobolev spaces $H^s(\mathbb{R})$ with $s\le\frac{3}{2}$ [27]. Consequently, the data-to-solution mapping for the equation (1.1) is not uniformly continuous in Sobolev spaces with exponent less than $\frac{3}{2}$.

For $k=2$ and $b=3$, the Eq. (1.1) becomes the Novikov equation with cubic nonlinearity

$$\partial_t u-\partial_t\partial_x^2 u+4u^2\partial_x u=3u\partial_x u\partial_x^2 u+u^2\partial_x^3 u, \tag{1.4}$$

which was discovered by V. Novikov in a symmetry classification of nonlocal partial differential equations with quadratic or cubic nonlinearity [42]. In [28], Himonas and Holliman considered the Cauchy problem of (1.4) in the Sobolev space $H^s(\mathbb{R})$ on the circle $\mathbb{T}$ and the line $\mathbb{R}$ for $s>\frac{3}{2}$. In [26], Grayshan investigated the non-periodic and the periodic low regularity solutions of the Eq. (1.4) in the Sobolev space with the exponent less than $\frac{3}{2}$. It is shown that the Eq. (1.4) admits an analytic solution if the initial data is analytic [47]. If the initial data satisfies a sign condition, Lai et at. [33] proved that the Eq. (1.4) admits a unique global weak solution in the Sobolev space $H^s(\mathbb{R})$ with $1\le s\le\frac{3}{2}$. In [39], Ni and Zhou considered the local well-posedness for the Eq. (1.4) in the Besov spaces $B^s_{2,r}(\mathbb{R})$ with the critical index $s=\frac{3}{2}$, and they also studied the well-posedness in $H^s(\mathbb{R})$ with $s>\frac{3}{2}$ by applying the Kato's semigroup theory. For the other works related to the Eq. (1.4), we refer the readers to [14,36,38,37,49] and the references therein.

The most celebrated integrable member of Eq. (1.1) is the following Camassa–Holm (CH) equation (for $k=1,b=2$) with quadratic nonlinearity,

$$\partial_t u-\partial_x^2\partial_t u+3u\partial_x u=2\partial_x u\partial_x^2 u+u\partial_x^3 u. \tag{1.5}$$

The CH equation is completely integrable, and it has a bi-Hamilton structure and infinite conservation laws as well as global dissipative solutions [4,8,31]. The CH equation has peakons which describe a fundamental characteristic of the traveling waves of largest amplitude, and these solutions can be formulated in the form of $ce^{-|x-ct|}$ with $c>0$ [13,16,18,9]. It is worth to mention that the CH equation leads to the geodesic flow of a certain invariant metric on the Bott–Virasoro group [15], which implies that the Least Action Principle remains to be true. The local well-posedness and blow-up phenomena of the CH equation in the Sobolev