J. Math. Anal. Appl. 445 (2017) 186–211

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The time viscosity-splitting method for the Boussinesq problem ☆

Tong Zhang [a,b,*], Yanxia Qian [a]

[a] *School of Mathematics & Information Science, Henan Polytechnic University, Jiaozuo, 454003, PR China*
[b] *Departamento de Matemática, Universidade Federal do Paraná, Centro Politécnico, Curitiba 81531-990, Brazil*

A R T I C L E I N F O

Article history:
Received 19 April 2016
Available online 3 August 2016
Submitted by Goong Chen

Keywords:
Boussinesq system
Viscosity-splitting scheme
Stability
Convergence

A B S T R A C T

In this article, we consider the time viscosity-splitting method for the Boussinesq problem. By using this technique, the considered problem is decoupled into two subproblems, and each subproblem is solved more easily than the original one. In the first step, two linear elliptic problems with semi-implicit schemes for the convection terms are solved. The advantage of using such numerical technique is that a linear system with the constant coefficient matrices is obtained and then the computation becomes easy. In the second step, a system consists of a Stokes problem and a linear elliptic problem is solved. The main results of this article include that (1) Establish the stability of numerical solutions in the time viscosity-splitting method; (2) Provide the convergence results of strongly second order for the velocity and temperature and strongly first order for the pressure. Finally, some numerical results are provided to display the performances of the developed numerical algorithms.

1. Introduction

The Boussinesq problem is a toy model to describe the convection phenomenon in the viscous incompressible flows and arises in the simplified models for geophysics (see [21]). For the mathematical analysis of the Boussinesq problem, we can refer to [12,30] and the references therein. In this paper, we present and analyze the time viscosity-splitting method for the Boussinesq problem with the external forcing functions. The considered Boussinesq problem consists of two coupled evolution equations, it can be described by the following equations:

☆ This work was supported by the NSF of China (No. 11301157), and the Foundation of Distinguished Young Scientists of Henan Polytechnic University (J2015-05), and the CAPES and CNPq of Brazil (No. 88881.068004/2014.01), and Natural Science Foundation of the Education Department of Henan Province of China (No. 14A110008).

* Corresponding author at: School of Mathematics & Information Science, Henan Polytechnic University, Jiaozuo, 454003, PR China.
E-mail address: tzhang@hpu.edu.cn (T. Zhang).

http://dx.doi.org/10.1016/j.jmaa.2016.07.023

$$\begin{cases} u_t(x,t) - \nu\Delta u(x,t) + (u(x,t)\cdot\nabla)u(x,t) + \nabla p(x,t) = -k\nu^2 j\theta(x,t) + f(x,t), & \text{in } \Omega\times(0,T],\\ \text{div } u(x,t) = 0, & \text{in } \Omega\times(0,T],\\ \theta_t(x,t) - \lambda\nu\Delta\theta(x,t) + (u(x,t)\cdot\nabla)\theta(x,t) = g(x,t), & \text{in } \Omega\times(0,T],\\ u(x,t) = 0,\ \theta(x,t) = 0, & \text{on } \Gamma\times(0,T],\\ u(x,0) = u_0, \theta(x,0) = \theta_0, & \text{on } \Omega\times\{0\}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary Γ. $u(x,t), p(x,t)$ and $\theta(x,t)$ are the fluid velocity, pressure and temperature at position $x \in \Omega$ and time $t \in (0,T]$ ($T > 0$ is the final time), respectively. $\nu > 0$ is the viscosity, k is the Groshoff number, $\lambda = Pr^{-1}$, Pr is the Prandtl number, $j = (1,0)^T$ is the vector of gravitational acceleration, ∇ is the gradient operator, Δ is the Laplacian operator, $f(x,t)$ and $g(x,t)$ are the forcing functions. When the variable temperature $\theta(x,t)$ is identically zero (or constant), the above system reduces to the classical incompressible Navier–Stokes equations.

Here and below, we consider a (regular) partition of $[0,T]$, $\triangle t = T/N$ and $t_n = n\triangle t$ $(n = 0,1,\cdots,N)$. In order to simplify the notations, we omit the space variables, i.e.

$$\varphi = \varphi(x,t) \ \text{ and } \ \varphi(t_n) = \varphi(x,t_n),$$

where φ may be u,θ,p,f and g. So is the same to the other notations in this paper.

In the last decades, much attention has been attracted for the problem (1.1). For the existence of the solutions for the Boussinesq problem, we can refer to the existence of time-periodic solutions [7,19] and traveling waves [8,18]. For the numerical schemes of the Boussinesq problem, we can refer to the standard Galerkin finite element method (FEM) [20], the projection-based stabilized mixed FEM [6], the variational multiscale method [9,35], the data assimilation algorithm [10,11] and the references therein. From the expression of the Boussinesq problem, we can see that problem (1.1) not only includes the incompressibility and strong nonlinearity, but also contains the coupling between the energy equation and the equations governing the fluid motion, so finding the numerical solutions of problem (1.1) becomes a very difficult task. If we solve the problem (1.1) directly, it means that we need to find the variables u,p and θ of (1.1) simultaneously, as a consequence, a large nonlinear discrete algebra system is formed. Generally speaking, it is expensive to find the numerical solutions of a such large coupled nonlinear system directly in standard Galerkin FEM. Therefore, the decoupled method was considered for the problem (1.1) in [37,38,33], and some meaningful results were established.

To our knowledge, one major difficulty for the numerical simulation of the incompressible flows is that the velocity and the pressure are coupled by the incompressibility constraint. The projection method is an efficient numerical scheme proposed by Chorin [4] and Temam [28]. This method is designed to consist of the projection of an intermediate velocity field onto the space of solenoidal vector fields, thus enforcing incompressibility. The most attractive feature of the projection method is that, at each time step, one only needs to solve a sequence of the decoupled linear elliptic equations for the different variables, then the computational scale is reduced and a lot of CPU time is saved. Therefore, the projection method has been widely studied in the incompressible flow. For example, we can refer to [1,3,22,31] for the Navier–Stokes equations, and [23,24] for the natural convection problem. Compared with above mentioned traditional projection method, in this paper, we develop a new projection method, the time viscosity-splitting scheme, to treat the Boussinesq problem (1.1). Our method is a two-step scheme that splits the nonlinearities and the incompressibility of the original problem into two different steps (but keeping the viscosity term, temperature term and the original boundary conditions in both steps). The time-discrete viscosity-splitting method can be described as follows.

(1) Given u^n and θ^n, find the intermediate approximations $(u^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}})$ of $(u(t_{n+1}),\theta(t_{n+1}))$. The intermediate velocity $u^{n+\frac{1}{2}}$ (as a first approximation of $u(t_{n+1})$) and the intermediate temperature $\theta^{n+\frac{1}{2}}$ (as a first approximation of $\theta(t_{n+1})$) are computed by a convection–diffusion problem.