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A singular perturbation result in competition theory ☆

Sergio Fernández-Rincón, Julián López-Gómez *

Departamento de Matemática Aplicada, Universidad Complutense de Madrid, Madrid 28040, Spain

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A B S T R A C T

This paper establishes a generalized version of the singular perturbation results given by V. Hutson et al. [10, Theorem 4.1] and X. He and W.M. Ni [6, Theorem 4.2 (iii)]. In particular, it ascertains the limiting profiles of the coexistence states of the classical Lotka–Volterra model for two competing species as the diffusion coefficients approximate zero. They are provided by the global attractors of the underlying non-spatial model whenever they exist.

1. Introduction

This paper studies the diffusive Lotka–Volterra competition model

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = \lambda(x)u - a(x)u^2 - b(x)uv & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial v}{\partial t} - d_2 \Delta v = \mu(x)v - c(x)uv - d(x)v^2 & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{in } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) = u_0 > 0, \quad v(\cdot, 0) = v_0 > 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\lambda, \mu, a, b, c, d \in \mathcal{C}(\bar{\Omega})$ with $b, c \geq 0$, $b \neq 0 \neq c$, and

$$\min_{\bar{\Omega}} a > 0 \quad \text{and} \quad \min_{\bar{\Omega}} d > 0.$$

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* Corresponding author. Fax: +34 91 394 41 02.
E-mail addresses: sergfern@ucm.es (S. Fernández-Rincón), julian@mat.ucm.es (J. López-Gómez).

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Thus, intra-specific competition occurs everywhere in $\bar{\Omega}$. In this model, non-flux boundary conditions are imposed for each of the species by assuming that $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$ for all $t > 0$, where ν stands for the outward unit normal vector field along the habitat edges. The functions u_0 and v_0 stand for the initial population densities in the habitat Ω; both are positive in the sense that they are non-negative and somewhere positive. Problem (1) and some simple variants of it have been dealt with, e.g., in [4,6,7,10–13].

The main result of this paper is a substantial generalization of [10, Theorem 4.1] that provides us with a sharp relation between the dynamics of (1) and the dynamics of the associated kinetic problem obtained by switching off to zero the diffusion coefficients d_1 and d_2 in (1), i.e.,

$$\begin{cases} \frac{\partial u}{\partial t} = \lambda(x)u - a(x)u^2 - b(x)uv & \text{in } (0,+\infty), \\ \frac{\partial v}{\partial t} = \mu(x)v - c(x)uv - d(x)v^2 & \text{in } (0,+\infty), \\ u(0) = u_0(x) \geq 0, \quad v(0) = v_0(x) \geq 0, \end{cases} \tag{2}$$

where $x \in \Omega$ is regarded as a sort of label. Throughout this paper, Problem (2) is refereed to as the *non-spatial* model. It admits three types of non-negative steady-state solutions: the *trivial* one, $(0,0)$, the two *semi-trivial* steady states,

$$(u,v) = \left(\tfrac{\lambda(x)}{a(x)}, 0\right) \quad \text{and} \quad (u,v) = \left(0, \tfrac{\mu(x)}{d(x)}\right),$$

whenever $\lambda(x) > 0$ and $\mu(x) > 0$, respectively, and the *coexistence* steady states, which acquire the form

$$(u,v) = \left(\tfrac{\lambda(x)d(x)-\mu(x)b(x)}{a(x)d(x)-b(x)c(x)}, \tfrac{\mu(x)a(x)-\lambda(x)c(x)}{a(x)d(x)-b(x)c(x)}\right),$$

provided both components are positive and $a(x)d(x) \neq b(x)c(x)$. For every $x \in \bar{\Omega}$, the global dynamics of (2) are determined by the existence and linearized stability of the semi-trivial steady-state solutions. Thus, $\bar{\Omega}$ can be divided into the next six regions:

- The area where both species become *extinct* as a result of the local attractive character of $(0,0)$ and the non-existence of any other component-wise non-negative steady-state solution of (2):

$$\Omega_{\text{ext}} := \left\{x \in \bar{\Omega} \;:\; \lambda(x) \leq 0 \;\text{ and }\; \mu(x) \leq 0\right\}. \tag{3}$$

- The *permanence* region, i.e., the open subset of $\bar{\Omega}$ where both semi-trivial steady states exist and are linearly unstable:

$$\Omega_{\text{per}} := \left\{x \in \bar{\Omega} \;:\; \lambda(x),\ \mu(x) > 0,\ \mu(x) > \tfrac{c(x)}{a(x)}\lambda(x),\ \lambda(x) > \tfrac{b(x)}{d(x)}\mu(x)\right\}. \tag{4}$$

  For every $x \in \Omega_{\text{per}}$, (2) possesses a unique coexistence steady state, which is a global attractor for all the component-wise positive solutions of the non-spatial model. This entails the *low competition* condition, $b(x)c(x) < a(x)d(x)$.

- The *bi-stability* region, which is the open subset of $\bar{\Omega}$ where both semi-trivial steady states exist and are linearly stable:

$$\Omega_{\text{bi}} := \left\{x \in \bar{\Omega} \;:\; \lambda(x),\ \mu(x) > 0,\ \mu(x) < \tfrac{c(x)}{a(x)}\lambda(x),\ \lambda(x) < \tfrac{b(x)}{d(x)}\mu(x)\right\}. \tag{5}$$

  For each $x \in \Omega_{\text{bi}}$, the *high competition* condition $b(x)c(x) > a(x)d(x)$ holds and hence, the model possesses a unique coexistence steady state which is a saddle point. Consequently, *founder control competition* occurs. This entails $b(x) > 0$ and $c(x) > 0$.