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Asymptotic profiles for the compressible Navier–Stokes equations in the whole space

Yoshiyuki Kagei [a], Masatoshi Okita [b]

[a] Faculty of Mathematics, Kyushu University, Nishi-ku, Motooka 744, Fukuoka 819-0395, Japan
[b] National Institute of Technology, Kurume College, Kurume-shi, Komorino 1-1-1, Fukuoka 830-8555, Japan

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ABSTRACT

This paper is concerned with the large time behavior of strong solutions of the compressible Navier–Stokes equation in the whole space $\mathbb{R}^n (n \geq 3)$ around the constant state. It was shown by Kawashima, Matsumura and Nishida (1979) and Hoff and Zumbrun (1995) that the perturbation of the constant state is time-asymptotic to a solution of the linearized problem, that is, a first-order asymptotic profile. In this paper a second-order asymptotic profile of the solution, which is caused by the nonlinear effect, is given.

1. Introduction

This paper studies the initial value problem for the compressible Navier–Stokes equation in $\mathbb{R}^n$, $n \geq 3$:

$$\begin{cases} \partial_t \rho + \operatorname{div} m = 0, \\ \partial_t m + \operatorname{div}\left(\frac{m \otimes m}{\rho}\right) + \nabla P(\rho) = \mu \Delta\left(\frac{m}{\rho}\right) + (\mu + \mu')\nabla \operatorname{div}\left(\frac{m}{\rho}\right), \\ (\rho, m)(0, x) = (\rho_0, m_0)(x). \end{cases} \tag{1}$$

Here $t > 0$, $x = {}^T(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, and the superscript ${}^T\cdot$ stands for the transposition; the unknown functions $\rho = \rho(t, x) > 0$ and $m = m(t, x) = {}^T(m_1(t, x), m_2(t, x), \cdots, m_n(t, x))$ denote the density and momentum, respectively; $P = P(\rho)$ is the pressure that is assumed to be a function of the density ρ; μ and μ' are the viscosity coefficients satisfying the conditions $\mu > 0$ and $\frac{2}{n}\mu + \mu' \geq 0$; and div, ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to x, respectively. The notation $\operatorname{div}\left(\frac{m \otimes m}{\rho}\right)$ means that its j-th component is given by $\operatorname{div}\left(\frac{m_j m}{\rho}\right)$.

We assume that $P(\rho)$ is smooth in a neighborhood of $\bar{\rho}$ with $P'(\bar{\rho}) > 0$, where $\bar{\rho}$ is a given positive constant.

E-mail address: m-okita@kurume-nct.ac.jp (M. Okita).

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In this paper we investigate asymptotic properties of strong solutions of the problem (1) around the constant stationary solution $(\bar{\rho}, 0)$.

Matsumura and Nishida [6] showed the global in time existence of the solution of (1) for $n = 3$, provided that the initial perturbation $u_0 = {}^T(\gamma(\rho_0 - \bar{\rho}), {}^T m_0)$, with $\gamma = \sqrt{P'(\bar{\rho})}$, is sufficiently small in $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Furthermore, the following decay estimates were obtained in [6]:

$$\|\nabla^k u(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad k = 0, 1, \tag{2}$$

where $u(t) = {}^T(\gamma(\rho - \bar{\rho}), {}^T m)$. (See also [7].) Kawashima, Matsumura and Nishida [4] proved that the solution is time asymptotic to the one of the linearized problem. It was shown in [4] that

$$\|u(t) - G(t) * u_0\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{1}{2}},$$

if u_0 is sufficiently small in $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Here $G(t) * u_0 = G(t, \cdot) * u_0$, where $G(t, x)$ is Green's matrix for the linearized system at ${}^T(\bar{\rho}, 0)$ for (1) and $*$ denotes the convolution with respect to x. Hoff and Zumbrun [3] derived a more detailed description of the large time behavior of $u(t)$ in L^p for all $1 \leq p \leq \infty$; the following estimates were established:

$$\|u(t)\|_{L^p} \leq C \begin{cases} (1+t)^{-\frac{n}{2}(1-\frac{1}{p})}, & 2 \leq p \leq \infty, \\ (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})} L(t), & 1 \leq p < 2, \end{cases}$$

and

$$\|u(t) - G(t) * u_0\|_{L^p} \leq C L(t) \begin{cases} (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}}, & 2 \leq p \leq \infty, \\ (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n}{4}(1-\frac{2}{p})-\frac{1}{2}+\eta}, & 1 \leq p < 2, \end{cases}$$

where η is any positive number. Here $L(t) = \log(1+t)$ when $n = 2$, and is otherwise identically one. Moreover, it was proved in [3] that $u(t)$ is time-asymptotic to the solution $\tilde{u}(t) = {}^T(\tilde{\sigma}, {}^T\tilde{m})$ of the following linear effective artificial viscosity system:

$$\begin{cases} \partial_t \tilde{\sigma} + \gamma \operatorname{div} \tilde{m} - \frac{1}{2}(\mu_2 + \mu_1)\Delta\tilde{\sigma} = 0, \\ \partial_t \tilde{m} - \mu_1 \Delta \tilde{m} - \frac{1}{2}(\mu_2 - \mu_1)\nabla \operatorname{div} \tilde{m} + \gamma \nabla \tilde{\sigma} = 0, \end{cases} \tag{3}$$

where $\mu_1 = \frac{\mu}{\bar{\rho}}$ and $\mu_2 = \frac{\mu + \mu'}{\bar{\rho}}$. More precisely, we have

$$\begin{aligned} &\|u(t) - \tilde{G}(t) * u_0\|_{L^p}, \ \|u(t) - \tilde{G}(t, \cdot)\int u_0 dx\|_{L^p} \\ &\leq C L(t) \begin{cases} (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}}, & 2 \leq p \leq \infty, \\ (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n}{4}(1-\frac{2}{p})-\frac{1}{2}+\eta}, & 1 \leq p < 2, \end{cases} \end{aligned}$$

for any positive constant η, where $\tilde{G}$ is Green's matrix for (3). We also mention that Kobayashi and Shibata [5] proved the following estimates

$$\|\partial_t^j \partial_x^\alpha G_1(t, x)\|_{L^p(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{j+|\alpha|}{2}}$$

for $2 \leq p \leq \infty$, and

$$\|\partial_t^j \partial_x^\alpha G_1(t, x)\|_{L^p(\mathbb{R}^n_x)} \leq C \begin{cases} (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{j+|\alpha|}{2}}, & n \geq 3 \text{ and } n\text{: odd}, \\ (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n}{4}(1-\frac{2}{p})-\frac{j+|\alpha|}{2}}, & n \geq 2 \text{ and } n\text{: even}, \end{cases}$$