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Spherical isometries of finite dimensional C^*-algebras

Ryotaro Tanaka [1]

Department of Mathematical Sciences, Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan

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ABSTRACT

In this paper, it is shown that every surjective isometry between the unit spheres of two finite dimensional C^*-algebras extends to a real-linear Jordan $*$-isomorphism followed by multiplication operator by a fixed unitary element. This gives an affirmative answer to Tingley's problem between two finite-dimensional C^*-algebras. Moreover, we show that if two finite dimensional C^*-algebras have isometric unit spheres, then they are $*$-isomorphic.

1. Introduction

Throughout this paper, all C^*-algebras are assumed to be unital. For a Banach space X, let $B(X)$ and $S(X)$ be the unit ball and unit sphere of X, respectively. This paper is concerned with the following problem.

Tingley's problem *(Tingley [21] 1987).* Let X and Y be Banach spaces. Suppose that $T_0 : S(X) \to S(Y)$ is a surjective isometry. Does there exist a real-linear isometric isomorphism $T : X \to Y$ satisfying $T|S(X) = T_0$.

The origin of Tingley's problem is the celebrated Mazur–Ulam theorem which states that every surjective isometry between normed spaces is automatically affine. This means, in a sense, that the (real) algebraic structure of a normed space is determined by its metric structure. Furthermore, in 1972, Mankiewicz [15] generalized the Mazur–Ulam theorem by showing that every surjective isometry between open connected subsets of real normed spaces is uniquely extended to an affine isometry between the whole spaces. In particular, a surjective isometry between the unit balls of two normed spaces extends to a real-linear isometric isomorphism. Inspired from this, Tingley [21] considered surjective isometries between unit spheres of normed space; and then asked whether or not they extend to real-linear isometries. Many papers have been devoted to studying Tingley's problem; see, e.g., [5,6,10] for recent development on the problem. However,

E-mail addresses: r-tanaka@math.kyushu-u.ac.jp, ryotarotanaka@m.sc.niigata-u.ac.jp.
[1] Current address: Faculty of Mathematics, Kyushu University, Fukuoka, 819-0395, Japan.

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Tingley's problem is still open even in the two-dimensional case. The survey of Ding [4] is one of the good starting points for understanding the problem.

There is another setting of Mazur–Ulam type problem important in this paper, that is, the case of unitary groups of C^*-algebras. In [8], Hatori and Molnár completely determined the forms of surjective isometries on the unitary group of the algebra of all bounded linear operators $\mathscr{B}(\mathscr{H})$ on a complex Hilbert space $\mathscr{H}$. From this result, in particular, it turned out that every surjective isometry on the unitary group of $\mathscr{B}(\mathscr{H})$ extends to a real-linear isometry. Moreover, in 2014, Hatori and Molnár [9] generalized their result by proving that every surjective isometry between the unitary groups of two von Neumann algebras extends to a real-linear isometry. What is interesting is that, although the Mazur–Ulam type problem on unitary groups can be viewed as a localization of Tingley's problem, the method used in [9] was completely different from those studied in the context of Tingley's problem. Actually, the proof of the result in [9] mentioned above is based on C^*-algebraic techniques such as Stone's theorem and the result of Kadison [11].

However, in the case of C^*-algebras, Tingley's problem can be closely related to the Mazur–Ulam type problem on unitary groups. Indeed, recently, it was shown in [20] that Tingley's problem has an affirmative answer for the case of $X = Y = \mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is finite dimensional (that is, $\mathscr{B}(\mathscr{H})$ is the algebra of all $n \times n$ complex matrices for some $n \in \mathbb{N}$). The solution is strongly based on the above mentioned result of Hatori and Molnár [8]; and C^*-algebraic methods are still effective for Tingley's problem on C^*-algebras.

The main purpose of this paper is to present, using both C^*-algebraic and Banach space geometric methods, a solution of Tingley's problem for the case of finite dimensional C^*-algebras. More precisely, it is shown that every surjective isometry between the unit spheres of two finite dimensional C^*-algebras extends to a real-linear Jordan $*$-isomorphism followed by multiplied by a fixed unitary element. Then, furthermore, we study the impact of the existence of surjective isometries between the unit spheres of two finite dimensional C^*-algebras. It turns out that if two finite dimensional C^*-algebras have isometric unit spheres, then they are $*$-isomorphic.

2. Extensions of spherical isometries

We start with the following basic result. The proof is based on Eidelheit's separation theorem [17, Theorem 2.2.26]; see, for example, [20] for the proof.

Lemma 2.1. *Let X be a Banach space. Suppose that C is a maximal convex subset of the unit sphere $S(X)$ of X. Then C is a norm exposed face of $B(X)$.*

We need the following result shown in [2, Lemma 5.1] (and [19, Lemma 3.5]).

Lemma 2.2. *Let X, Y be Banach spaces, and let $T : S(X) \to S(Y)$ be a surjective isometry. Then C is a maximal convex subset of $S(X)$ if and only if $T(C)$ is that of $S(Y)$.*

Let $\mathscr{R}$ be a von Neumann algebra. As was shown in [7, Theorem 5.3] (see also [1, Theorem 4.4]), every weak-operator closed face F of $B(\mathscr{R})$ is associated with a (unique) partial isometry $V \in \mathscr{R}$ under the equation

$$F = V + (1 - VV^*)B(\mathscr{R})(1 - V^*V) = \{A \in B(\mathscr{R}) : AV^* = VV^*\}.$$

In particular, if $\mathfrak{A}$ is a finite dimensional C^*-algebra, it can be viewed as a von Neumann algebra (by considering any faithful representation). Hence each norm closed (hence compact) face of $B(\mathfrak{A})$ has such a form.

The following is a key ingredient for our main result.