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On the Matsumoto–Yor property in free probability

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ABSTRACT

We study the Matsumoto–Yor property in free probability. We prove that the limiting empirical eigenvalue distribution of the GIG matrices and the Marchenko–Pastur distribution have the free Matsumoto–Yor property. Finally we characterize these distributions by regression properties in free probability.

1. Introduction

This paper is devoted to the study of free probability analogues of so called Matsumoto–Yor property. In [15] Matsumoto and Yor noted an interesting independence property of Generalized Inverse Gaussian (GIG) and Gamma distributions, namely if X has $GIG(-p, a, b)$ distribution, where $p, a, b > 0$ and Y has Gamma distribution $G(p, a)$ then

$$U = \frac{1}{X+Y} \qquad \text{and} \qquad V = \frac{1}{X} - \frac{1}{X+Y}$$

are independent and distributed $GIG(-p, b, a)$ and $G(p, b)$ respectively.

By a Generalized Inverse Gaussian distribution $GIG(p, a, b)$, $p \in \mathbb{R}$, $a, b > 0$ we understand here a probability measure given by the density

$$f(x) = \frac{1}{K(p,a,b)} x^{p-1} e^{-ax-\frac{b}{x}} I_{(0,+\infty)}(x),$$

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where $K(p, a, b)$ is a normalization constant. A Gamma distribution $G(p, a)$, $p, a > 0$ is given by the density

$$g(x) = \frac{a^p}{\Gamma(p)} x^{p-1} e^{-ax} I_{(0,+\infty)}(x).$$

In [13] Letac and Wesołowski proved that the above independence property characterizes GIG and Gamma laws, i.e. for independent X and Y random variables U and V defined as above are independent if and only if X has $GIG(-p, a, b)$ distribution and Y has $G(p, a)$ distribution, for some $p, a, b > 0$. In the same paper authors proved similar result for real symmetric matrices.

It is remarkable that many characterizations of probability measures by independence properties have parallel results in free probability. A basic example of such analogy is Bernstein's theorem which says that for independent random variables X and Y, random variables $X+Y$ and $X-Y$ are independent if and only if X and Y are Gaussian distributed. A free probability analogue proved in [16] gives a characterization of Wigner semicircle law by similar properties where the independence assumptions are replaced by freeness assumptions. Another well studied example of such analogy is Lukacs theorem which characterizes the Gamma law by independence of $X + Y$ and $X/(X + Y)$ for independent X and Y (cf. [14]). The free analogue (see [4,20]) turns out to characterize a Marchenko–Pastur law which is also known as the free Poisson distribution.

In this paper we will be particularly interested in characterizations of probability measures by regression assumptions. Well known example of such characterization is provided by so called Laha–Lukacs regressions [12] which characterize Meixner distributions by assumption that for independent X and Y the first conditional moment of X given by $X+Y$ is a linear function of $X+Y$ and the second conditional moment of the same type is a quadratic function of $X+Y$. In [4,6] authors studied free analogues of Laha–Lukacs regressions, which characterize free Miexner distribution defined in [2]. This result motivated a more intensive study of regression characterizations in free probability (see [7,8,21]).

The aim of this paper is to settle a natural question whether also the Matsumoto–Yor property possesses a free probability analogue. The answer turns out to be affirmative. We prove a free analogue of the regression version of this property proved in [25], where instead of assuming independence of U and V, constancy of regressions of V and V^{-1} given U is assumed. The role of the Gamma law is taken as in the case of Lukacs theorem by the Marchenko–Pastur distribution. The free probability analogue of the GIG distribution turns out to be a measure defined in [9], where it appears as a limiting empirical spectral distribution of complex GIG matrices. The proof of the main result partially relies on the technique which we developed in our previous papers [21,19], however some new ideas were needed. As a side product we prove that the free GIG and the Marchenko–Pastur distributions have a free convolution property which mimics the convolution property of the classical GIG and Gamma distributions.

The paper is organized as follows, in Section 2 we give a brief introduction to free probability and we investigate some basic properties of the free GIG distribution, Section 3 is devoted to proving that the Marchenko–Pastur and the free GIG distributions have the freeness property which is an analogue of the classical Matsumoto–Yor property. Section 4 contains the main result of this paper which is a regression characterization of the Marchenko–Pastur and the free GIG distributions.

2. Preliminaries

In this section we will give a short introduction to free probability theory, which is necessary to understand the rest of the paper. Next we give the definitions of the Marchenko–Pastur and the free GIG distribution. Since the free GIG appears for the first time in the context of free probability, we study some basic properties of this distribution.