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Optimal Gabor frame bounds for separable lattices and estimates for Jacobi theta functions

Markus Faulhuber[a,*], Stefan Steinerberger[b]

[a] NuHAG, Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
[b] Department of Mathematics, Yale University, 10 Hillhouse Avenue, New Haven, CT 06510, USA

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ABSTRACT

We study sharp frame bounds of Gabor frames for integer redundancy with the standard Gaussian window and prove that the square lattice optimizes both the lower and the upper frame bound among all rectangular lattices. This proves a conjecture of Floch, Alard & Berrou (as reformulated by Strohmer & Beaver). The proof is based on refined log-convexity/concavity estimates for the Jacobi theta functions θ_3 and θ_4.

1. Introduction

1.1. Introduction

The study of Gabor frames originates in a 1946 paper of Gabor [10] in which he describes intermediate cases between pure time analysis and pure frequency analysis (Fourier analysis). He proposes to have a two-dimensional representation of a one-dimensional signal (function) which simultaneously uses information of the distribution of the signal in time and its frequencies. A Gabor system (or Weyl–Heisenberg system) for $L^2(\mathbb{R}^d)$ is generated by a (fixed, non-zero) window function $g \in L^2(\mathbb{R}^d)$ and an index set $\Lambda \subset \mathbb{R}^{2d}$ and is denoted by $\mathcal{G}(g, \Lambda)$. It consists of time-frequency shifted versions of g. We say $\lambda = (x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$ is a point in the time-frequency plane and use the following notation for a time-frequency shift by λ

$$\pi(\lambda)g(t) = M_\omega T_x\, g(t) = e^{2\pi i \omega \cdot t} g(t - x), \quad x, \omega, t \in \mathbb{R}^d.$$

* Corresponding author.
E-mail addresses: markus.faulhuber@univie.ac.at (M. Faulhuber), stefan.steinerberger@yale.edu (S. Steinerberger).

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Hence, for a window function g and an index set Λ the Gabor system is

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g \mid \lambda \in \Lambda\}.$$

The time-frequency shifted versions of the window g are called atoms. In order to be a frame, $\mathcal{G}(g, \Lambda)$ has to satisfy the frame inequality

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^d)$$

for some positive constants $0 < A \leq B < \infty$ called frame constants or frame bounds. Whenever we speak of frame bounds, we only consider the optimal frame bounds. The frame bounds serve as quantitative measurement of how close the frame is to a tight frame, in which case we would have $A = B$. If the frame gives rise to an orthonormal basis, we have $A = B = 1$. The index set $\Lambda \subset \mathbb{R}^{2d}$ is called a lattice if it is generated by an invertible (non-unique) $2d \times 2d$ matrix S, in the sense that $\Lambda = S\mathbb{Z}^{2d}$. The volume of the lattice, which is unique, is defined as

$$\mathrm{vol}(\Lambda) = |\det(S)| \qquad \text{while its density or redundancy is given by} \qquad \delta(\Lambda) = \frac{1}{\mathrm{vol}(\Lambda)}.$$

A lattice is called separable if the generating matrix can take the form

$$S = \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix}.$$

For more details on frames, Gabor frames and time-frequency analysis we refer to the classical texts [3,6–8,11,14]. One of the fundamental questions in Gabor analysis is to understand when a Gabor system $\mathcal{G}(g, \Lambda)$ forms a frame. For a fixed window g the family of all lattices Λ which together with g generate a frame is called the frame set of the window g. We distinguish between the full frame set whose elements are lattices in general and the reduced frame set whose elements are the lattice parameters of separable lattices [12]. For a window function $g \in L^2(\mathbb{R}^d)$ we denote the full frame set by

$$\mathcal{F}_{full}(g) = \{\Lambda \subset \mathbb{R}^{2d} \ \text{ lattice} \mid \ \mathcal{G}(g, \Lambda) \text{ is a frame}\}$$

and the reduced frame set by

$$\mathcal{F}_{(\alpha,\beta)}(g) = \{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \ \mathcal{G}(g, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d) \text{ is a frame}\}.$$

Clearly, $(\alpha, \beta) \in \mathcal{F}_{(\alpha,\beta)}(g)$ implies $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d \in \mathcal{F}_{full}(g)$. We may rephrase the question about when a Gabor system forms a frame in the following way. For any given g, what is its (full or reduced) frame set? At this point we want to emphasize that there is no general idea of how to determine the frame set of a class of functions or even a single function. Even less is known about how the frame bounds change within the frame set. The 1-dimensional standard Gaussian window $g_0(t) = 2^{1/4}e^{-\pi t^2}$ has been fully analyzed: results of Lyubarskii [17] and Seip [19] give the full frame set for Gabor frames with a Gaussian window g as

$$\mathcal{F}_{full}(g) = \{\Lambda \subset \mathbb{R}^2 \mid \mathrm{vol}(\Lambda) < 1\}.$$

This implies that the reduced frame set is given by

$$\mathcal{F}_{(\alpha,\beta)}(g) = \{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \alpha\beta < 1\}.$$

However, it is still not clear how to find the lattice that optimizes the frame bounds.