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Ideal invariant injections

Marek Balcerzak, Szymon Głąb [1], Jarosław Swaczyna *

Institute of Mathematics, Łódź University of Technology, ul. Wólczańska 215, 93-005 Łódź, Poland

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ABSTRACT

For an ideal $\mathcal{I}$ on ω, we introduce the notions of $\mathcal{I}$-invariant and bi-$\mathcal{I}$-invariant injections from ω to ω. We study injections that are invariant with respect to selected classes of ideals. We show some applications to ideal convergence.

1. Introduction

Let $\omega := \{0, 1, \dots\}$, $\mathbb{Z}$ stands for the set of all integers, and id is the identity function on ω. By *an ideal* $\mathcal{I}$ *on* ω we mean an ideal of subsets of ω such that $\omega \notin \mathcal{I}$ and $\{n\} \in \mathcal{I}$ for all $n \in \omega$. If $\mathcal{I}$ is an ideal on ω then $\mathcal{I}^\star$ denotes its dual filter $\{\omega \setminus A \colon A \in \mathcal{I}\}$. Several examples of ideals on ω were considered in [8] (see also [16,18] and [14]). The ideal of all finite subsets of ω is denoted by Fin.

Through the paper, we will work with injections from ω to ω. The set of all such injections will be denoted by **Inj**. Fix an ideal $\mathcal{I}$ on ω and let $f \in \mathbf{Inj}$. We say that f is *$\mathcal{I}$-invariant* if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. We say that f^{-1} is *$\mathcal{I}$-invariant* if $f^{-1}[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. If f and f^{-1} are $\mathcal{I}$-invariant then f is called *bi-$\mathcal{I}$-invariant*. Note that every $f \in \mathbf{Inj}$ is bi-Fin-invariant.

We start from easy facts and simple examples.

Fact 1. *Let $\mathcal{I}$ be an ideal on ω and let $f \in \mathbf{Inj}$.*

(i) *f^{-1} is $\mathcal{I}$-invariant if and only if $f[A] \notin \mathcal{I}$ for every $A \notin \mathcal{I}$.*
(ii) *If $f[\omega] \in \mathcal{I}$ then f is $\mathcal{I}$-invariant and it is not bi-$\mathcal{I}$-invariant.*

* Corresponding author.
E-mail addresses: marek.balcerzak@p.lodz.pl (M. Balcerzak), szymon.glab@p.lodz.pl (S. Głąb), jswaczyna@wp.pl (J. Swaczyna).

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Proof. (i) "$\Rightarrow$" Let $A \notin \mathcal{I}$ and suppose that $f[A] \in \mathcal{I}$. Then $A = f^{-1}[f[A]] \in \mathcal{I}$, a contradiction.

"$\Leftarrow$" Assume that $f[A] \notin \mathcal{I}$ for every $A \notin \mathcal{I}$. Suppose that f^{-1} is not $\mathcal{I}$-invariant. Hence $f^{-1}[B] \notin \mathcal{I}$ for some $B \in \mathcal{I}$. Then $B \supseteq f[f^{-1}[B] \notin \mathcal{I}$, a contradiction.

(ii) The first part is obvious, and the second part follows from $f^{-1}[f[\omega]] = \omega \notin \mathcal{I}$. $\square$

To show an example based on Fact 1(ii), recall the definition of the *classical density ideal* $\mathcal{I}_d$. For a set $A \subseteq \omega$, consider the following numbers

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{|A \cap \{0,\dots,n-1\}|}{n}, \quad \overline{d}(A) := \limsup_{n\to\infty} \frac{|A \cap \{0,\dots,n-1\}|}{n}.$$

If $\underline{d}(A) = \overline{d}(A)$, we denote this common value by $d(A)$ and call the *asymptotic density* of A. Then define $\mathcal{I}_d := \{A \subseteq \omega \colon \overline{d}(A) = 0\}$.

Note that every increasing injection is $\mathcal{I}_d$-invariant. In particular, $f(n) := n^2$ is $\mathcal{I}_d$-invariant. Moreover, in this case $f[\omega] \in \mathcal{I}_d$. Hence f is not bi-$\mathcal{I}_d$-invariant by Fact 1(ii).

The next example shows an ideal $\mathcal{I}$ on ω and a bijection f from ω onto ω which is $\mathcal{I}$-invariant but f^{-1} is not so. If $k, l \in \omega$ and $k > 0$, we denote $k\omega + l := \{kn + l \colon n \in \omega\}$.

Example 2. Let $f \colon \omega \to \omega$ be given by the formulas: $f(2n) := 4n$, $f(4n+1) = 4n+2$, $f(4n+3) := 2n+1$ for $n \in \omega$. Then f is a bijection. Consider the ideal $\mathcal{I}$ defined as follows

$$\mathcal{I} := \{A \cup B \colon A \in \mathrm{Fin},\ B \subseteq 2\omega\}.$$

Clearly, f is $\mathcal{I}$-invariant. Note that $4\omega + 1 \notin \mathcal{I}$ but $f[4\omega + 1] \in \mathcal{I}$. Let $B := f[4\omega + 1]$. Then $B \in \mathcal{I}$ and $f^{-1}[B] \notin \mathcal{I}$. $\square$

An ideal $\mathcal{I}$ on ω is called *tall* if every infinite subset of ω contains an infinite set belonging to $\mathcal{I}$ (see [8]). Note that the ideal in the Example 2 is not tall. The respective example with a tall ideal will be presented in Section 4.

For $f \colon \omega \to \omega$ let $\mathrm{Fix}(f) := \{n \in \omega \colon f(n) = n\}$. The following fact is obvious.

Fact 3. *Let $\mathcal{I}$ be an ideal on ω and $f \in \mathbf{Inj}$. If $\mathrm{Fix}(f) \in \mathcal{I}^\star$ then f is bi-$\mathcal{I}$-invariant.*

The purpose of our paper is to describe $\mathcal{I}$-invariant and bi-$\mathcal{I}$-invariant injections for selected classes of ideals. In some cases, we also study topological features of the sets of such injections. It is easy to see that $\mathbf{Inj}$ is a G_δ subset of the Baire space ω^ω (cf. [24, p. 66]), so it is a Polish space, by the Alexandrov theorem. Sets of the form $\{f \in \mathbf{Inj} \colon f(k_i) = l_i \text{ for } i = 1,\dots,p\}$ constitute a base of the topology in $\mathbf{Inj}$. We are interested in the Baire category and levels of the Borel hierarchy for the sets of $\mathcal{I}$-invariant and bi-$\mathcal{I}$-invariant injections in the space $\mathbf{Inj}$.

Proposition 4. *The set $\{f \in \mathbf{Inj} \colon \omega \setminus \mathrm{Fix}(f) \in \mathrm{Fin}\}$ is dense in $\mathbf{Inj}$. In particular, the set $\{f \in \mathbf{Inj} : f$ is bi-$\mathcal{I}$-invariant$\}$ is dense in $\mathbf{Inj}$ for every ideal $\mathcal{I}$ containing all singletons. Moreover, if $\mathcal{I}$ contains infinite sets and all singletons, the set $\{f \in \mathbf{Inj} : f$ is not $\mathcal{I}$-invariant$\}$ is dense in $\mathbf{Inj}$ as well.*

Proof. Let $V := \{f \in \mathbf{Inj} \colon f(k_i) = l_i \text{ for } i = 1,\dots,p\}$ be a basic set. To prove the first assertion, define $g : \omega \to \omega$ as follows. Pick $n \in \omega$ such that $k_i \le n$ and $l_i \le n$ for $i = 1,\dots,p$. Put $g(k_i) := l_i$ for $i = 1,\dots,p$ and extend g on $\{0,\dots,n\}$ to be a bijection of this set onto itself. Finally put $g(k) := k$ for $k > n$. Then $g \in V$ and $\omega \setminus \mathrm{Fix}(g) \in \mathrm{Fin}$. Next use Fact 3.

To prove the second assertion, fix distinct $k_1,\dots,k_p \in \omega$ and distinct $l_1,\dots,l_p \in \omega$. Set $B := \{k_1,\dots,k_p,l_1,\dots l_p\}$ and consider a bijection h from B onto itself that $h_1(k_i) = l_i$ for $i \in \{1,\dots,p\}$.