J. Math. Anal. Appl. 445 (2017) 443–458

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Weighted Procrustes problems

Maximiliano Contino [a], Juan Ignacio Giribet [a,b,*], Alejandra Maestripieri [a,b]

[a] *Facultad de Ingeniería, Universidad de Buenos Aires, Paseo Colón 850, (1063) Buenos Aires, Argentina*
[b] *Instituto Argentino de Matemática "Alberto P. Calderón", CONICET, Saavedra 15, Piso 3, (1083) Buenos Aires, Argentina*

ARTICLE INFO

Article history:
Received 10 December 2015
Available online 28 July 2016
Submitted by J.A. Ball

Keywords:
Operator approximation
Schatten p classes
Oblique projections

ABSTRACT

Let $\mathcal{H}$ be a Hilbert space, $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$ and $W \in L(\mathcal{H})$ a positive operator such that $W^{1/2}$ is in the p-Schatten class, for some $1 \leq p < \infty$. Given $A \in L(\mathcal{H})$ with closed range and $B \in L(\mathcal{H})$, we study the following weighted approximation problem: analyze the existence of

$$\min_{X \in L(\mathcal{H})} \|AX - B\|_{p,W},$$

where $\|X\|_{p,W} = \|W^{1/2}X\|_p$. In this paper we prove that the existence of this minimum is equivalent to a compatibility condition between $R(B)$ and $R(A)$ involving the weight W, and we characterize the operators which minimize this problem as W-inverses of A in $R(B)$.

1. Introduction

One problem of interest in Signal and Image Processing is to find low dimensional models that approximate, in some sense, given data [11,16]. In particular many of these problems can be posed as follows: given a matrix $B \in \mathbb{C}^{n\times n}$, with $\operatorname{rank}(B) \geq k$, for $k \in \mathbb{N}$ satisfying $k < n$, find a matrix $Y_0 \in \mathbb{C}^{n\times n}$ with $\operatorname{rank}(Y_0) = k$ such that,

$$Y_0 = \underset{\{Y \in \mathbb{C}^{n\times n}:\ \operatorname{rank}(Y)=k\}}{\operatorname{argmin}} f(Y - B),$$

for some *cost* function $f : \mathbb{C}^{n\times n} \to \mathbb{R}$. Due to its intractability, usually this problem is studied by relaxating the constraint on the rank of Y, which under certain conditions, turns out to be an exact relaxation. For this, the factorization $Y = AX$ is used, with $A \in \mathbb{C}^{n\times k}$, $X \in \mathbb{C}^{k\times n}$. Assume that the cost function is given by the Frobenius norm $\|\cdot\|_F$, now we are interested in the following problem:

* Corresponding author at: Facultad de Ingeniería, Universidad de Buenos Aires, Paseo Colón 850, (1063) Buenos Aires, Argentina.
E-mail addresses: mcontino@fi.uba.ar (M. Contino), jgiribet@fi.uba.ar (J.I. Giribet), amaestri@fi.uba.ar (A. Maestripieri).

http://dx.doi.org/10.1016/j.jmaa.2016.07.050
0022-247X/

$$Y_0 = \underset{\{X\in\mathbb{C}^{k\times n},\ A\in\mathbb{C}^{n\times k}\}}{\operatorname{argmin}} \|AX - B\|_F. \tag{1.1}$$

In fact, suppose that $A_0 \in \mathbb{C}^{n\times k}$, $X_0 \in \mathbb{C}^{k\times n}$ satisfy

$$\|A_0X_0 - B\|_F = \min_{\{X\in\mathbb{C}^{k\times n},\ A\in\mathbb{C}^{n\times k}\}} \|AX - B\|_F, \tag{1.2}$$

then,

$$\|A_0X_0 - B\|_F = \min_{X\in\mathbb{C}^{k\times n}} \|A_0X - B\|_F. \tag{1.3}$$

If a (positive) weight is introduced in equation (1.3), or if the Frobenius norm is replaced by another unitary invariant norm, the same problem can be studied.

This work is devoted to study an extension of problem (1.3) in abstract Hilbert spaces. More specifically, we study the following approximation problem. Given $A \in L(\mathcal{H})$ with closed range, $B \in L(\mathcal{H})$ and $W \in L(\mathcal{H})$ a positive operator, we analyze the conditions for the existence of

$$\min_{X\in L(\mathcal{H})} \|W^{1/2}(AX - B)\|_p, \tag{1.4}$$

for $1 \le p < \infty$, where $\|\cdot\|_p$ is the p-Schatten norm.

There are several examples of these minimization problems in Control Theory and Signal Processing [12,29]. Similar problems also arise in Quantum Chemistry, for example in the orthogonalization process of Löwdin [2,21], or in the approximation of the Hamiltonian operator [17,18,23].

The existence of minimum of $\|AX - B\|_p$ in Hilbert spaces, was studied in [22] using differentiation techniques and also in [15], where a connection between p-Schatten norms and the order in $L(\mathcal{H})^+$ (the cone of semidefinite positive operators) is established. However, the introduction of a weight $W \in L(\mathcal{H})^+$ plays an important role, since we are introducing on $\mathcal{H}$ a semi-inner product associated to W for which $\mathcal{H}$ is no longer a Hilbert space, unless W is invertible. In this case, the existence of a suitable orthogonal projection is not guaranteed. In fact the existence of a W-orthogonal projection onto $R(A)$ depends on the relationship between the weight W and the closed subspace $R(A)$.

The notion of compatibility, defined in [8] and developed later in [6,9,10], has its origin in the work of Z. Pasternak-Winiarski [26]. In that work the author studied, for a fixed subspace $\mathcal{S}$, the analyticity of the map $W \to P_{W,\mathcal{S}}$ which associates to each positive invertible operator W the orthogonal projection onto $\mathcal{S}$ under the (equivalent) inner product $\langle x, y\rangle_W = \langle Wx, y\rangle$, for $x, y \in \mathcal{H}$. The notion of compatibility appears when W is allowed to be any positive semidefinite operator, not necessarily invertible (and even, a selfadjoint bounded linear operator). More precisely, W and $\mathcal{S}$ are said to be *compatible* if there exists a (bounded linear) projection Q with range $\mathcal{S}$ which satisfies $WQ = Q^*W$. If W is positive and invertible or $\mathcal{H}$ has finite dimension, there exists a unique projection onto $\mathcal{S}$ which is W-selfadjoint [8]. In general, it may happen that there is no such Q or that there is an infinite number of them. However, there exists an angle condition between $\mathcal{S}^\perp$ and $\overline{W(\mathcal{S})}$ which determines the existence of these projections [13]. In fact, the existence of such projections is related with the existence of minimum of equation (1.4).

The contents of the paper are the following. In section 2, some characterizations of the compatibility of the pair $(W, R(A))$ are given. Also some properties of shorted operators and compressions and its connection with compatibility are stated. Finally, the concept of W-inverses of an operator A in the range of an operator B, and some properties are presented.

For the sake of simplicity, in section 3, we study problem (1.4) when $B = I$. We prove that the infimum of the set $\{(AX - I)^*W(AX - I) : X \in L(\mathcal{H})\}$ (where the order is the one induced by the cone of positive operators), always exists and is equal to $W_{/R(A)}$, the shorted operator of W to $R(A)$. We also prove that