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A uniqueness result for a semipositone p-Laplacian problem on the exterior of a ball

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A B S T R A C T

We consider steady state reaction diffusion equations on the exterior of a ball, namely, boundary value problems of the form:

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u) & \text{in } \Omega_E, \\ u = 0 & \text{on } |x| = r_0, \\ u \to 0 & \text{when } |x| \to \infty, \end{cases}$$

where $\Delta_p z := \operatorname{div}(|\nabla z|^{p-2}\nabla z)$, $1 < p < n$, λ is a positive parameter, $r_0 > 0$ and $\Omega_E := \{x \in \mathbb{R}^n \mid |x| > r_0\}$. Here the weight function $K \in C^1[r_0, \infty)$ satisfies $K(r) > 0$ for $r \geq r_0$, $\lim_{r\to\infty} K(r) = 0$, and the reaction term $f \in C[0,\infty) \cap C^1(0,\infty)$ is strictly increasing and satisfies $f(0) < 0$ (semipositone), $\limsup_{s\to 0+} sf'(s) < \infty$, $\lim_{s\to\infty} f(s) = \infty$, $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$ and $\frac{f(s)}{s^q}$ is nonincreasing on $[a, \infty)$ for some $a > 0$ and $q \in (0, p-1)$. For a class of such steady state equations it turns out that every nonnegative radial solution is strictly positive in the exterior of a ball, and exists for $\lambda \gg 1$. We establish the uniqueness of this positive radial solution for $\lambda \gg 1$.

1. Introduction

Study of semipositone problems, namely, the analysis of positive solutions to boundary value problems of the form:

$$\begin{cases} -\Delta_p u = \lambda \tilde{f}(u) & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases}$$

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where $\Delta_p z := \mathrm{div}(|\nabla z|^{p-2}\nabla z)$, $p > 1$, λ is a positive parameter, D is a bounded domain in $\mathbb{R}^n$, $n > 1$ and $\tilde{f} \in C[0,\infty)$ with $\tilde{f}(0) < 0$, $\tilde{f}$ nondecreasing and $\lim_{s\to\infty} \tilde{f}(s) = \infty$, has been of significant interest in the mathematical community for the past twenty five to thirty years. It has been well documented in the history that the analysis of such nonlinear eigenvalue problems for positive solutions when $\tilde{f}(0) < 0$ is very challenging. The focus in this paper is in the case when $\lim_{s\to\infty} \frac{\tilde{f}(s)}{s^{p-1}} = 0$ ($p-1$ sublinear growth at infinity). For this case, for existence results for $\lambda \gg 1$, see [2,10] when $p = 2$, and [8,9,13] when $p > 1$. For uniqueness results, see [1,11] when D is a ball and $p = 2$, [3] when D is a bounded domain and $p = 2$, and [8] when D is a ball and $p > 1$. When D is a bounded domain and $p > 1$, this question on uniqueness still remains unsettled.

Recently, this study of semipositone problems has been considered on domains exterior to a ball when $1 < p < n$. Namely, to problems of the form:

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u) & \text{in } \Omega_E, \\ u = 0 & \text{on } |x| = r_0, \\ u \to 0 & \text{when } |x| \to \infty, \end{cases} \tag{1}$$

where $r_0 > 0$ and $\Omega_E := \{x \in \mathbb{R}^n \mid |x| > r_0\}$. Here $f \in C[0,\infty) \cap C^1(0,\infty)$ and $K \in C^1[r_0,\infty)$ satisfy:

(H_1) $f(0) < 0$ (semipositone),
(H_2) $\limsup_{s\to 0+} sf'(s) < \infty$,
(H_3) $f' > 0$ on $(0,\infty)$ and $f(s) \to \infty$ as $s \to \infty$,
(H_4) $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$,
(H_5) $K(r) > 0$ for $r \geq r_0$ and there exist $\tilde{d} > 0$ and $\sigma \in (0, \frac{n-p}{p-1})$ such that $K(r) \leq \frac{\tilde{d}}{r^{n+\sigma}}$ for $r \gg 1$.

In particular, see [12,15] where the existence of positive radial solutions was established for $\lambda \gg 1$ (see also [6] where this study has been extended without restricting to the case of radial solutions).

The main goal of this paper is to establish the uniqueness of this radial solution when $\lambda \gg 1$ under the following additional assumptions:

(H_6) there exist $q \in (0, p-1)$ and $a > 0$ such that $\frac{f(s)}{s^q}$ is nonincreasing on $[a,\infty)$,
(H_7) $r^{\frac{p(n-1)}{p-1}} K(r)$ is strictly increasing on $[r_0,\infty)$.

Namely, we prove:

Theorem 1.1. *Assume* (H_1)–(H_7) *hold. Then* (1) *has a unique positive radial solution for* $\lambda \gg 1$.

We note that by using the change of variables $r = |x|$ and $t = \left(\frac{r}{r_0}\right)^{\frac{n-p}{1-p}}$, study of (1) can be reduced to analyzing the two point boundary value problem of the form:

$$\begin{cases} -\left(\varphi_p(u'(t))\right)' = \lambda h(t) f(u(t)), & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{2}$$

where $\varphi_p(s) := |s|^{p-2}s$ and $h(t) := \left(\frac{p-1}{n-p}\right)^p r_0^p t^{\frac{p(1-n)}{n-p}} K\left(r_0 t^{\frac{1-p}{n-p}}\right)$. Then $h \in C^1(0,1]$ since $K \in C^1[r_0,\infty)$. By (H_5) and (H_7) we have that h is strictly decreasing, $\underline{h} := \inf_{t\in(0,1]} h(t) > 0$ and there exist $d > 0$ and