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Carleson measures for Hilbert spaces of analytic functions on the complex half-plane

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ABSTRACT

The notion of a *Carleson measure* was introduced by Lennart Carleson in his proof of the Corona Theorem for $H^\infty(\mathbb{D})$. In this paper we will define it for certain type of reproducing kernel Hilbert spaces of analytic functions of the complex half-plane, $\mathbb{C}_+$, which will include Hardy, Bergman and Dirichlet spaces. We will obtain several necessary or sufficient conditions for a positive Borel measure to be Carleson by preforming tests on reproducing kernels, weighted Bergman kernels, and studying the tree model obtained from a decomposition of the complex half-plane. The Dirichlet space will be investigated in detail as a special case. Finally, we will present a control theory application of Carleson measures in determining admissibility of controls in well-posed linear evolution equations.

1. Introduction

Let μ be a positive Borel measure on a set $\Omega \subseteq \mathbb{C}$, and let $\mathcal{H}$ be a Hilbert space of complex-valued functions on Ω. If there exists a constant $C(\mu)$, depending on μ only, such that for all $h \in \mathcal{H}$ we have

$$\int_\Omega |h|^2 \, d\mu \leq C(\mu) \|h\|_{\mathcal{H}}^2, \tag{1}$$

then μ is called a *Carleson measure* for $\mathcal{H}$ and we shall refer to (1) as the *Carleson criterion*. The set of Carleson measures for $\mathcal{H}$ will be denoted by $CM(\mathcal{H})$. The notion of a Carleson measure was introduced by Lennart Carleson in his proof of the Corona Theorem for $H^\infty(\mathbb{D})$ in [5], where a complete characterisation of Carleson measures for $H^p(\mathbb{D})$ $(1 \leq p < \infty)$ was given. In 1967 Lars Hörmander extended Carleson's result to the unit ball of $\mathbb{C}^n$ [17], and since then many other generalisations and variants of this idea have been studied (we mention in particular the characterisation of Carleson measures for the weighted Bergman spaces on $\mathbb{D}$ by J. Cima and W. Wogen in [6] and on the unit ball of $\mathbb{C}^n$ by D. Luecking in [22], and for

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the weighted Dirichlet space on $\mathbb{D}$ by D. Stegenga in [25]). The popularity of this area of research is a consequence of wide applicability of Carleson embeddings, going far beyond Carleson's original formulation of this concept, and in particular their usefulness in the study of certain classes of operators acting on $\mathcal{H}$ (for example multiplication operators [21,25]). However, this area of research is usually limited to the case of $\Omega = \mathbb{D}$ or the unit ball of $\mathbb{C}^n$, and other domains are rarely considered.

In this paper we shall consider

$$\Omega = \mathbb{C}_+ := \{z = x + iy \in \mathbb{C} \; : \; x > 0\},$$

the open right complex half-plane. This choice of domain is not arbitrary and its motivation is drawn from two main reasons. First of all, for some of the most well-known Hilbert spaces of analytic functions on the open unit disk of the complex place, such as the Hardy space H^2 [8,23], the Bergman space $\mathcal{B}^2$ [10,15] or the Dirichlet space $\mathcal{D}$ [3,11], there exists a fundamental relation between the norm on each of this spaces and the norm of some weighted sequence space ℓ^2, namely

$$\|f\|^2_{H^2} := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{i\theta})\right|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 \qquad \left(\forall f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2\right),$$

$$\|f\|^2_{\mathcal{B}^2} := \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 \, dz = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} \qquad \left(\forall f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}^2\right),$$

$$\|f\|^2_{\mathcal{D}} := \|f\|^2_{H^2} + \|f'\|^2_{\mathcal{B}^2} = \sum_{n=0}^{\infty} (n+1)|a_n|^2 \qquad \left(\forall f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}\right).$$

But of course for some problems it is more natural to consider the continuous version of the weighted sequence space ℓ^2, that is the weighted $L^2(0, \infty)$ space. It follows from the Plancherel's Theorem, that for some class of weights, the Laplace transform ($\mathfrak{L}$) is an isometric map from the weighted space of square-(Lebesgue)-integrable functions on the positive real half-line to some spaces of analytic functions defined on the open right complex half-plane (we shall present this statement rigorously in the next section). And for example, if we denote by $H^2(\mathbb{C}_+)$, $\mathcal{B}^2(\mathbb{C}_+)$ and $\mathcal{D}(\mathbb{C}_+)$ the spaces of Hardy, Bergman and Dirichlet (respectively) on the half-plane, we have:

$$\|F\|^2_{H^2(\mathbb{C}_+)} := \sup_{x>0} \int_{-\infty}^{\infty} |F(x+iy)|^2 \, \frac{dy}{2\pi} = \int_0^{\infty} |f(t)|^2 \, dt,$$

$$\|F\|^2_{\mathcal{B}^2(\mathbb{C}_+)} := \int_{\mathbb{C}_+} |F(z)|^2 \, \frac{dz}{\pi} = \int_0^{\infty} |f(t)|^2 \, \frac{dt}{t},$$

$$\|F\|^2_{\mathcal{D}(\mathbb{C}_+)} := \|F\|^2_{H^2(\mathbb{C}_+)} + \|F'\|^2_{\mathcal{B}^2(\mathbb{C}_+)} = \int_0^{\infty} |f(t)|^2 (t+1) \, dt,$$

for all $F = \mathfrak{L}[f]$ in $H^2(\mathbb{C}_+)$, or in $\mathcal{B}^2(\mathbb{C}_+)$, or in $\mathcal{D}(\mathbb{C}_+)$ and f in an appropriate weighted L^2 space on $(0, \infty)$.

One of the instances where the continuous setting is more suitable, and also the second reason motivating our research of Carleson measures for these spaces, is the study of control and observation operators for linear evolution equations. It has been shown in [20] that the admissibility criterion for these operators is equivalent to the Carleson criterion. We shall explain it in the concluding section of this paper.