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Asymptotically autonomous multivalued Cauchy problems with spatially variable exponents

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ABSTRACT

We study the asymptotic behavior of a non-autonomous multivalued Cauchy problem of the form

$$\frac{\partial u}{\partial t}(t) - \operatorname{div}(D(t)|\nabla u(t)|^{p(x)-2}\nabla u(t)) + |u(t)|^{p(x)-2}u(t) + F(t,u(t)) \ni 0$$

on a bounded smooth domain Ω in $\mathbb{R}^n$, $n \geq 1$ with a homogeneous Neumann boundary condition, where the exponent $p(\cdot) \in C(\overline{\Omega})$ satisfies $p^- := \min p(x) > 2$. We prove the existence of a pullback attractor and study the asymptotic upper semicontinuity of the elements of the pullback attractor $\mathfrak{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ as $t \to \infty$ for the non-autonomous evolution inclusion in a Hilbert space H under the assumptions, amongst others, that F is a measurable multifunction and $D \in L^\infty([\tau,T]\times\Omega)$ is bounded above and below and is monotonically nonincreasing in time. The global existence of solutions is obtained through results of Papageorgiou and Papalini.

1. Introduction

In this paper we study a multivalued Cauchy problem of the form

$$\begin{cases} \frac{\partial u}{\partial t}(t) - \operatorname{div}(D(t)|\nabla u(t)|^{p(x)-2}\nabla u(t)) + |u(t)|^{p(x)-2}u(t) + F(t,u(t)) \ni 0 \\ u(\tau) = u_0 \end{cases} \tag{1}$$

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on a bounded smooth domain Ω in $\mathbb{R}^n$, $n \geq 1$ with a homogeneous Neumann boundary condition, where the exponent $p(\cdot) \in C(\overline{\Omega})$ satisfies

$$p^+ := \max_{x\in\overline{\Omega}} p(x) \geq p^- := \min_{x\in\overline{\Omega}} p(x) > 2,$$

and the initial condition is $u(\tau) \in H := L^2(\Omega)$. The terms D and F are assumed to satisfy:

Assumption D. $D : [\tau, T] \times \Omega \to \mathbb{R}$ is a function in $L^\infty([\tau, T] \times \Omega)$ satisfying:
(D1) There are positive constants, β and M such that $0 < \beta \leq D(t, x) \leq M$ for almost all $(t, x) \in [\tau, T] \times \Omega$.
(D2) $D(t, x) \geq D(s, x)$ for each $x \in \Omega$ and $t \leq s$ in $[\tau, T]$.

Assumption F. $F : [\tau, T] \times H \to P_f(H)$, where

$$P_f(H) := \{A \subset H : A \text{ nonempty and closed}\}$$

is a multifunction satisfying:
(F1) For all $x \in H$, $t \mapsto F(t, x)$ is measurable, that is, for all $y \in H$, the function

$$[\tau, T] \ni t \mapsto d(y, F(t, x)) := \inf\{\|y - z\|_H : z \in F(t, x)\} \in \mathbb{R}$$

is measurable.
(F2) There exists $k \in L^1([\tau, T], \mathbb{R}^+)$ such that

$$h(F(t, x), F(t, y)) \leq k(t)\|x - y\|,$$

a.e. on $[\tau, T]$, for all $x, y \in H$. Here h denotes the Hausdorff metric on $P_f(H)$ given by: for $A, B \in P_f(H)$,

$$h(A, B) := \max\{\text{dist}(A, B), \text{dist}(B, A)\},$$

where $\text{dist}(A, B) := \sup\{d(a, B) : a \in A\}$, $d(a, B) := \inf\{\|a - b\|_H : b \in B\}$ and similarly for $\text{dist}(B, A)$.
(F3) There exist $a, c \in L^2([\tau, T], \mathbb{R}^+)$:

$$\|F(t, x)\| := \sup\{\|z\|_H : z \in F(t, x)\} \leq a(t) + c(t)\|x\|_H,$$

a.e. in $[\tau, T]$, for all $x \in H$.

In [9] the authors proved the existence of a pullback attractor for the following non-autonomous evolution equation

$$\frac{\partial u}{\partial t}(t) - \text{div}(D(t)|\nabla u(t)|^{p(x)-2}\nabla u(t)) + |u(t)|^{p(x)-2}u(t) = B(t, u(t)) \tag{2}$$

on a bounded smooth domain Ω in $\mathbb{R}^n$, B was globally Lipschitz in its second variable and D was assumed to satisfy **Assumption D** above. Moreover, they proved upper semicontinuity of pullback attractors when the diffusion parameters vary. A similar problem was studied in [15] for a constant exponent p and stronger conditions on the diffusion coefficients D. In [10], the authors considered $B(t, u(t)) \equiv B(u)$ in the problem (2) and proved that it is asymptotically autonomous.

The paper is organized as follows. In Section 2 we prove existence of solution for inclusion (1) following Papageorgiou and Papalini [13]. In Section 3 we provide estimates on the solutions. In Section 4 we establish the existence of a pullback attractor. In Section 5 we prove the asymptotic upper semicontinuity of the elements of the pullback attractor, i.e., we prove that the inclusion (1) is, in fact, asymptotically autonomous when F does not depend explicitly on t.