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Besov and Triebel–Lizorkin regularity for the Hodge decomposition and applications to magnetic potentials

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ABSTRACT

The Hodge decomposition is a fundamental result in the theory of differential forms. It has been extensively studied from different perspectives and it is the source of several applications in physics and mathematics. For example, it is important in the analysis of Navier–Stokes' and Maxwell's equations, it can be used to prove results on inverse scattering problems in quantum mechanics and it is useful for the study of magnetic potentials associated to a magnetic field. The regularity properties of the Hodge decomposition have been studied in several works in the last decades, most prominently it has been proved that Sobolev regularity holds true. In this paper we prove regularity for the Hodge decomposition in a much larger scale of function spaces: Besov and Triebel–Lizorkin spaces on compact Riemannian manifolds with boundary. We prove regularity in such spaces for the Hodge–Morrey decomposition and the Hodge–Morrey–Friedrichs decomposition. We, furthermore, state and prove a refinement for the Hodge–Morrey–Friedrichs decomposition (with regularity) that allows expressing every differential form as the sum of the differential of a form plus the co-differential of another form plus a harmonic field belonging to a finite dimensional space. This holds true even if the original form is barely differentiable (with sth order of differentiability, for every real number $s > 0$). We finally provide an application to the study of magnetic potentials associated to magnetic fields. More precisely, we prove the existence of a magnetic potential associated to a magnetic field. The former being one time more differentiable than the latter, in terms of Besov and Triebel–Lizorkin regularity. This is an important motivation for us, since we have the intention to apply our results to further investigations in relativistic inverse (time dependent) scattering problems in quantum mechanics, where Besov and Triebel–Lizorkin regularity for magnetic potentials is relevant.

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1. Introduction

The Hodge decomposition asserts that any differential form is the sum of an exact form, a co-exact form and a harmonic field. More precisely: We denote by M a Riemannian manifold and by $\Omega^k(M)$ the space of infinitely differentiable differential forms. For any form $\omega \in \Omega^k(M)$ there exist differential forms a_ω, b_ω and γ_ω such that

$$\omega = da_\omega + \delta b_\omega + \gamma_\omega,$$

where d is the differential operator, δ is the co-differential operator (the dual operator of d in the sense of distributions) and γ_ω is a harmonic field, i.e. $d\gamma_\omega = 0$, $\delta\gamma_\omega = 0$. The harmonic field can be chosen satisfying certain boundary conditions that make the vector space of such harmonic fields finite dimensional.

If ω is closed ($d\omega = 0$), one can prove that $b_\omega = 0$ and we have

$$\omega = da_\omega + \gamma_\omega.$$

Thus, we can express any closed form as the sum of an exact form plus a harmonic field. The elements of the finite-dimensional vector space of harmonic fields represent the closed differential forms that are not exact. They give information from the topology of the manifold M, since they represent classes of the de Rham cohomology of M.

The Hodge decomposition is discussed in several references, the classical ones being [20] and [25]. It is generalized to k-forms in Sobolev spaces defined in compact Riemannian manifolds with boundary. This is known as the Hodge–Morrey–Friedrichs decomposition on forms with Sobolev-Class (see [32]). There are also sharp Hodge decompositions on forms in Sobolev–Besov classes for bi- and tridimensional Lipschitz domains in certain compact Riemaniann manifolds without boundary [27,26]. In this paper we present the Hodge Decomposition for k-forms with Besov and Triebel–Lizorkin class in a compact Riemannian manifold with boundary. We, furthermore, state and prove a refinement for the Hodge–Morrey–Friedrichs decomposition (with regularity) that allows expressing every differential form as the sum of the differential of a form plus the co-differential of another form plus a harmonic field belonging to a finite dimensional space. This holds true even if the original form is barely differentiable (with sth order of differentiability, for every real number $s > 0$). We additionally provide an application to the study of magnetic potentials associated to magnetic fields. More precisely, we prove the existence of a magnetic potential associated to a magnetic field. The former being one time more differentiable than the latter, in terms of Besov and Triebel–Lizorkin regularity.

The Hodge decomposition is a very useful tool to analyse some partial differential equations (see [11], for example) and it has also important applications to physics. It is a fundamental ingredient for the understanding of the Aharonov–Bohm effect (see [3–7]). It is also related with properties of solutions of differential equations in connection with topological aspects. The Hodge decomposition plays a fundamental role in the projection methods, which are adopted in the numerical solution of the Navier–Stokes equations for incompressible flows [11].

Regularity properties of the Hodge decomposition also play an important role in applications (see [3–7]). They can be used to determine some regularity features of solutions of certain differential equations (see for example [19,3,28,37]). There are many works that study regularity for the Hodge decomposition. In [32] Sobolev spaces are addressed. Hölder regularity (with differentiation parameter $\lambda \in (0, 1)$) is one main goal of the doctoral thesis of Bolik [8], where an explicit formula for the components of the decomposition is found in terms of Green functions. In the present work we study Besov and Triebel–Lizorkin properties for the Hodge decomposition. We analyze the scales of function spaces B^s_{pq} (Besov spaces) and F^s_{pq} (Triebel–Lizorkin spaces) for $s > 0$ and $2 \le p, q \le \infty$ – for the Triebel–Lizorkin spaces we additionally require $p < \infty$.

The Besov and Triebel–Lizorkin spaces constitute a very wide class of function spaces that contain in some sense most of the possible spaces concerning a notion of differentiability. These include the Lipschitz