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Lipschitz-type conditions on homogeneous Banach spaces of analytic functions

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ABSTRACT

In this paper we deal with Banach spaces of analytic functions X defined on the unit disk satisfying that $R_t f \in X$ for any $t > 0$ and $f \in X$, where $R_t f(z) = f(e^{it}z)$. We study the space of functions in X such that $\|P_r(Df)\|_X = O(\frac{\omega(1-r)}{1-r})$, $r \to 1^-$ where $Df(z) = \sum_{n=0}^{\infty}(n+1)a_n z^n$ and ω is a continuous and non-decreasing weight satisfying certain mild assumptions. The space under consideration is shown to coincide with the subspace of functions in X satisfying any of the following conditions: (a) $||R_t f - f||_X = O(\omega(t))$, (b) $||P_r f - f||_X = O(\omega(1-r))$, (c) $||\Delta_n f||_X = O(\omega(2^{-n}))$, or (d) $||f - s_n f||_X = O(\omega(n^{-1}))$, where $P_r f(z) = f(rz)$, $s_n f(z) = \sum_{k=0}^{n} a_k z^k$ and $\Delta_n f = s_{2^n} f - s_{2^{n-1}} f$. Our results extend those known for Hardy or Bergman spaces and power weights $\omega(t) = t^\alpha$.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ be the Fréchet space of all analytic functions in the unit disk $\mathbb{D}$ endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$. For $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $0 \le r < 1$ we write $P_r f$ and $R_t f$ for the dilation and rotation operators, i.e. for $0 \le r < 1$ and $t \in \mathbb{R}$

$$P_r f(z) = f(rz) \text{ and } R_t f(z) = f(e^{it}z).$$

As usual, we use the notation $s_n f = \sum_{k=0}^{n} a_k z^k$, $\Delta_n f = s_{2^n} f - s_{2^{n-1}} f$ and $\sigma_n f = \sum_{k=0}^{n}(1 - \frac{k}{n+1})a_k z^k$ for each $f \in \mathcal{H}(\mathbb{D})$.

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A Banach space X is said to be a *Banach space of analytic functions* (called $\mathcal{H}$-admissible in [3]) if

$$A(\mathbb{D}) \subset X \subset \mathcal{H}(\mathbb{D}),$$

with continuous inclusions, where $A(\mathbb{D})$ stands for the disk algebra.

We shall write $\mathcal{P}$ for the subspace of polynomials and we shall denote by $X_{\mathcal{P}}$ the closure of $\mathcal{P}$ under the norm in X, i.e. $\overline{\mathcal{P}} = X_{\mathcal{P}}$ or equivalently $\overline{A(\mathbb{D})} = X_{\mathcal{P}}$. Of course $X_{\mathcal{P}}$ is also a Banach space of analytic functions and

$$X_{\mathcal{P}} \subseteq \{f \in X : \lim_{t\to 0^+} ||R_t f - f||_X = 0\}. \tag{1.1}$$

A Banach space of analytic functions X is said to be *homogeneous* (see [3, Definition 4.1]) whenever X also satisfies the following properties

$$f \in X \Longrightarrow R_t f \in X \text{ and } ||R_t f||_X = ||f||_X \text{ for every } t \in [0, 2\pi), \tag{1.2}$$

$$f \in X \Longrightarrow P_r f \in X \text{ and } ||P_r f||_X \le K||f||_X \text{ for every } 0 \le r < 1, \tag{1.3}$$

for some absolute constant $K \ge 1$.

Most of the classical spaces such as Hardy spaces H^p, Bergman spaces A^p, $BMOA$, the Bloch space $\mathcal{B}$ and many others are homogeneous spaces of analytic functions (see [6], [8] or [15]).

A basic fact holding for homogeneous spaces to be used in the sequel is that for each $f \in X$ the map $w \to f_w$, where $f_w(z) = f(wz)$ for $w \in \overline{\mathbb{D}}$ defines an $X_{\mathcal{P}}$-valued analytic function i.e. $F(w) = f_w \in \mathcal{H}(\mathbb{D}, X_{\mathcal{P}})$. In particular

$$M_X(r, f) := \sup_{|w|=r} ||f_w||_X$$

is an increasing function of r and $M_X(r, f) = ||P_r f||_X$.

Moreover the function F belongs to $A(\mathbb{D}, X_{\mathcal{P}})$, the space of all vector-valued bounded holomorphic functions $F : \mathbb{D} \to X_{\mathcal{P}}$ with continuous extension to the boundary equipped with the norm

$$||F||_{A(\mathbb{D},X_{\mathcal{P}})} = \sup_{|w|\le 1} ||F(w)||_X = \sup_{|\zeta|=1} ||F(\zeta)||_X = \|f\|_X.$$

Of course if X is a homogeneous Banach space of analytic functions, so it is $X_{\mathcal{P}}$. Actually, for homogeneous Banach spaces of analytic functions, (1.3) together with the fact that $P_r f \in A(\mathbb{D})$ for each $0 \le r < 1$ and polynomials are dense in $A(\mathbb{D})$ allow us to characterize $X_{\mathcal{P}}$ as

$$X_{\mathcal{P}} = \{f \in X : \lim_{r\to 1^-} ||P_r f - f||_X = 0\}. \tag{1.4}$$

The study of the subspace of $X_{\mathcal{P}}$ with a fixed rate of convergence to zero in (1.1) goes back to the work of Hardy and Littlewood in the twenties for the case $X = H^p$. Their fundamental contribution, proved in a series of papers ([9,10] and [11]), can be condensed in the following result.

Theorem (H-L). *Let $1 \le p < \infty$, $0 < \alpha \le 1$ and $f \in H^p$. Then the following statements are equivalent:*

(*a*) $||R_t f - f||_{H^p} = O(t^\alpha)$, $t \to 0^+$,
(*b*) $M_{H^p}(r, f') = O((1 - r)^{\alpha-1})$, $r \to 1^-$.