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Center boundaries for planar piecewise-smooth differential equations with two zones

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ABSTRACT

This paper is concerned with 1-parameter families of periodic solutions of piecewise smooth planar vector fields, when they behave like a center of smooth vector fields. We are interested in finding a separation boundary for a given pair of smooth systems in such a way that the discontinuous system, formed by the pair of smooth systems, has a continuum of periodic orbits. In this case we call the separation boundary as a *center boundary*. We prove that given a pair of systems that share a hyperbolic focus singularity p_0, with the same orientation and opposite stability, and a ray Σ_0 with endpoint at the singularity p_0, we can find a smooth manifold Ω such that $\Sigma_0 \cup \{p_0\} \cup \Omega$ is a center boundary. The maximum number of such manifolds satisfying these conditions is five. Moreover, this upper bound is reached.

1. Introduction

One of the most challenging problems in the qualitative theory of planar ordinary differential equations is the second part of the classical 16th Hilbert problem: the determination of an upper bound for the number of limit cycles for the class of polynomial vector fields of degree n. This problem remains unsolved if $n \geq 2$. The case $n = 1$ has a trivial answer because we can not have limit cycles for linear systems. By the other hand, we can have limit cycles for planar piecewise linear differential systems. It means that this problem, in the context of piecewise smooth systems, has attracted much attention.

The study of piecewise linear differential systems goes back to Andronov and coworkers [1]. These systems are used to model many real processes and different modern devices, see for more details [2] and the references therein.

The simplest case of piecewise linear differential systems is the one in which we have two half-planes separated by a straight line W. If both linear vector fields coincide at each point $w \in W$ we say that it

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is the case of continuous piecewise linear differential systems. In 1990, Lum and Chua conjectured that a continuous piecewise linear vector field in the plane with two zones has at most one limit cycle, see [14]. In 1998 this conjecture was proved by Freire, Ponce, Rodrigo and Torres in [8].

In the literature we can find a lot of works that deal with limit cycles of discontinuous piecewise linear differential systems, see for instance [5,6,9–12]. Han and Zang, in [10], provide discontinuous systems with two limit cycles, and they conjecture that the maximum number of limit cycles for this class is exactly two. However, in [11], Huan and Yang presented numerical analysis showing that an example with three limit cycles could exist. Later on, Llibre and Ponce provide in [12] a proof of the existence of such three limit cycles. In [5] the authors obtain three limit cycles from a piecewise perturbation of a linear center, and they can choose from which periodic orbits of the linear center the limit cycles bifurcate. To the best of our knowledge, we do not know an example of planar piecewise linear systems separated by a straight line W with four or more limit cycles.

In planar piecewise linear differential systems, the separation boundary W between the two zones plays an important role. After using some broken line as the boundary between linear zones, Braga and Mello in [3] put in evidence the important role of the separation boundary in determining the number of limit cycles. In [3] the authors exhibit an example with seven limit cycles having W as a polygonal curve and they state the conjecture: "*Given $n \in \mathbb{N}$ there is a piecewise linear system with two zones in the plane with exactly n limit cycles*". This conjecture was proved by the same authors in the paper [4]. The main idea of [13] for three zones is still valid when only two zones exist. Novaes and Ponce in [15] gave another solution to the Braga–Mello Conjecture.

In the paper [4], the authors consider piecewise linear systems sharing a singular point of focus type, both in the Jordan Normal Form. In Lemma 2 of [4] is proved that there exists a piecewise linear separation boundary such that the discontinuous system has a center. Our work is inspired in this result. We are interested in find separation boundary for a given pair of piecewise smooth systems in such a way that the discontinuous system has a continuum of periodic solutions. In this case we call the separation boundary as a *center boundary*. In [15] the center boundary considered is the y-axis, this choice is possibly due to the special nature of the eigenvalues of the system chosen. Here in this work we discuss the case of piecewise linear systems when we have two foci not necessarily in the Jordan Normal Form. We deal not only with piecewise linear systems but also with piecewise smooth systems not necessarily linear. Our work is also related with stability issues in the active field of switched control systems, see for instance [16,17,19].

We consider pairs of differential systems of class $\mathcal{C}^r$, $r \geq 2$, in the following way. Let $U \subset \mathbb{R}^2$ be an open set and consider $f_1, f_2 : U \to \mathbb{R}^2$ of class $\mathcal{C}^r$. We denote the pair of differential systems

$$\dot{X} = f_1(X), \tag{1}$$

and

$$\dot{X} = f_2(X), \tag{2}$$

by $Z = (f_1, f_2)$. The set of all pairs $Z = (f_1, f_2)$ of systems (1) and (2) we denote by $\mathfrak{X}^r$. For each $X_0 \in U$, $i = 1, 2$, we denote the solution of $\dot{X} = f_i(X)$ that passes through X_0 at $t = 0$ by γ_i, i.e., $\gamma_i(0, X_0) = X_0$.

Let $W \subset U$ be a piecewise smooth manifold in such a way that the set $U \setminus W$ has two connected components, i.e., $U = U_1 \cup U_2 \cup W$, where U_1 and U_2 are connected open sets. Given the pair $Z = (f_1, f_2)$ we define $Z_W = (f_1, f_2, W)$ as a piecewise smooth differential system

$$\dot{X} = \begin{cases} f_1(X) \text{ if } X \in U_1 \cup W, \\ f_2(X) \text{ if } X \in U_2 \cup W. \end{cases} \tag{3}$$

System (3) is accept to be multi-valued at W.