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Asymptotic stability and bifurcation of time-periodic solutions for the viscous Burgers' equation

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A B S T R A C T

We consider the Dirichlet boundary value problem for the viscous Burgers' equation with a time periodic force on a one dimensional finite interval. Under the boundedness assumption on the external force, we prove the existence of the time-periodic solution by using the Galerkin method and Schaefer's fixed point theorem. Furthermore, we show that this time-periodic solution is unique and time-asymptotically stable in the H^1 sense under an additional smallness condition on the external force. It is naturally expected that when the amplitude of the external force increases and crosses a certain critical value, the time-periodic solution is no longer asymptotically stable. In the last part of the article, to support our theory, numerical experiments are carried out to investigate the exchange of stabilities of the time-periodic solutions when the amplitude of the force crosses the first critical value. We numerically find this critical value at which the stable solutions turn into the unstable ones.

Contents

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1. Introduction

In this article, we prove the existence, uniqueness and stability of the time-periodic solutions to the viscous Burgers' equation in a one dimensional bounded domain:

$$u_t - \nu u_{xx} + u u_x = F(x,t), \quad x \in (0,1), \quad t \geq 0, \tag{1.1}$$

where $\nu > 0$ is the viscosity coefficient and $F(x,t)$ is a time-periodic external force with period T, i.e., $F(x,t+T) = F(x,t)$ for $t \geq 0$, $x \in [0,1]$.

Furthermore, the Dirichlet boundary condition is imposed:

$$u(0,t) = u(1,t) = 0, \ t \geq 0. \tag{1.2}$$

In this setting, we seek the time-periodic solution with a period T, i.e.,

$$u(x,t) = u(x,t+T), \ x \in [0,1], \ t \geq 0. \tag{1.3}$$

The viscous Burgers' equation is one of the most fundamental partial differential equations in fluid dynamics, and it has been served as a simple model equation derived from the Navier–Stokes equations to study certain features of the fluid motion. The inviscid Burgers' equation, i.e. $\nu = 0$, is a model equation describing nonlinear transport effect. More specifically, the inviscid solutions are the nonlinear waves propagated along the characteristic curves determined by the ordinary differential equation $dx/dt = u$. From this observation, one can easily see that the equation may develop the singularity, called the shock, in a finite time no matter how smooth the initial data is given. This model equation is often used to explain how the shock waves are generated in compressible gas dynamics.

In [14,33] the authors investigated the convergence of the viscous Burgers' solutions to the inviscid ones, i.e. the limit when the viscosity ν tends to zero. The vanishing viscosity problem is one of the outstanding problems in fluid dynamics, see, e.g., [4,5,34,18,20,21,15]. In general, we observe the sharp transition layers along the characteristics of the inviscid equations. Incorporating the so-called singular perturbation analysis the sharp transition layers like boundary and interior layers are understood and accordingly the asymptotic behaviors at the small viscosity ν of the viscous flow of the Navier–Stokes or Burgers type have been tackled.

For the viscous Burgers' equation, one of the most interesting features is the presence of the viscous shock waves that are regularized traveling wave solutions associated with the shock waves. More specifically, the viscous Burgers equation admits the traveling front solution connecting the two different constant states by which its traveling speed s is determined by the Rankine–Hugoniot condition. The stability and the asymptotic behavior of this front solutions have been studied intensively in several different settings. See [3,29,8,10].

Although the Burgers' equation is fundamental, relatively little has been known about the time-periodic solutions to (1.1)–(1.2). The physical models where the time-periodic external force influences the system often arise in practical situations. For examples, among numerous others, periodic motion of objects, synchronization of oscillations can be taken. This is a main motivation of the study of the time-periodic problems. It is thus natural to question whether the system or equation mathematically admits the corresponding time-periodic solutions or not. Indeed, many authors concentrate on the existence and uniqueness of time-periodic solutions, see [7,2,1,27,23,9,24,16,25,22,28,13,6,30,32,12,17,19]. Furthermore, if exists, their stability is an interesting issue, i.e., it is interesting to know how the solutions near the time-periodic solutions behave as time goes on. Under the smallness assumption of the external force, we indeed prove in this article the global H^1 asymptotic stability in the main theorem below (Theorem 1.1). For the convenience of the readers, using the technique of the Galerkin methods we give a proof of the existence of the time-periodic solutions. Finally, the numerical evidence supports the H^1 global stability and shows the bifurcation diagram (Fig. 2).