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Verblunsky coefficients related with periodic real sequences and associated measures on the unit circle ☆

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ABSTRACT

It is known that given a pair of real sequences $\{\{c_n\}_{n=1}^{\infty}, \{d_n\}_{n=1}^{\infty}\}$, with $\{d_n\}_{n=1}^{\infty}$ a positive chain sequence, we can associate a unique nontrivial probability measure μ on the unit circle. Precisely, the measure is such that the corresponding Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$ are given by the relation

$$\alpha_{n-1} = \overline{\rho}_{n-1}\left[\frac{1-2m_n-ic_n}{1-ic_n}\right], \quad n \geq 1,$$

where $\rho_0 = 1$, $\rho_n = \prod_{k=1}^{n}(1-ic_k)/(1+ic_k)$, $n \geq 1$ and $\{m_n\}_{n=0}^{\infty}$ is the minimal parameter sequence of $\{d_n\}_{n=1}^{\infty}$. In this paper we consider the space, denoted by N_p, of all nontrivial probability measures such that the associated real sequences $\{c_n\}_{n=1}^{\infty}$ and $\{m_n\}_{n=1}^{\infty}$ are periodic with period p, for $p \in \mathbb{N}$. By assuming an appropriate metric on the space of all nontrivial probability measures on the unit circle, we show that there exists a homeomorphism g_p between the metric subspaces N_p and V_p, where V_p denotes the space of nontrivial probability measures with associated p-periodic Verblunsky coefficients. Moreover, it is shown that the set F_p of fixed points of g_p is exactly $V_p \cap N_p$ and this set is characterized by a $(p-1)$-dimensional submanifold of $\mathbb{R}^p$. We also prove that the study of probability measures in N_p is equivalent to the study of probability measures in V_p. Furthermore, it is shown that the pure points of measures in N_p are, in fact, zeros of associated para-orthogonal polynomials of degree p. We also look at the essential support of probability measures in the limit periodic case, i.e., when the sequences $\{c_n\}_{n=1}^{\infty}$ and $\{m_n\}_{n=1}^{\infty}$ are limit periodic with period p. Finally, we give some examples to illustrate the results obtained.

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1. Introduction

Given a nontrivial probability measure $\mu(z) = \mu(e^{i\theta})$ on the unit circle $\mathbb{T} = \{z = e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, the associated sequence of orthogonal polynomials on the unit circle (OPUC, in short) $\{\phi_n(z)\}_{n=0}^{\infty}$ is that with the property

$$\int_{\mathbb{T}} \bar{z}^j \phi_n(z) d\mu(z) = \int_0^{2\pi} e^{-ij\theta} \phi_n(e^{i\theta}) d\mu(e^{i\theta}) = 0, \quad 0 \leq j \leq n-1, \quad n \geq 1,$$

where $\phi_0(z) = 1$. Letting $\kappa_n^{-2} = \|\phi_n\|^2 = \int_{\mathbb{T}} |\phi_n(z)|^2 d\mu(z)$, the orthonormal polynomials on the unit circle are $\varphi_n(z) = \kappa_n \phi_n(z)$, $n \geq 0$.

The polynomials $\phi_n(z)$, $n \geq 0$ (assumed here as monic polynomials) satisfy the relations

$$\begin{aligned} \phi_n(z) &= z\phi_{n-1}(z) - \overline{\alpha}_{n-1}\, \phi_{n-1}^*(z), \\ \phi_n(z) &= (1 - |\alpha_{n-1}|^2) z\phi_{n-1}(z) - \overline{\alpha}_{n-1} \phi_n^*(z), \end{aligned} \qquad n \geq 1,$$

where $\alpha_{n-1} = -\overline{\phi_n(0)}$ and $\phi_n^*(z) = z^n \overline{\phi_n(1/\bar{z})}$ denotes the reversed (reciprocal) polynomial of $\phi_n(z)$. The complex numbers α_n, in recent years, have been referred to as Verblunsky coefficients. It is known that these coefficients are such that $|\alpha_n| < 1$, $n \geq 0$. On the other hand, Verblunsky's Theorem shows that given any sequence of complex numbers with modulus less than one there exists a unique associated nontrivial probability measure on the unit circle (see [18, Theorem 1.7.11]). Therefore, the OPUC and the associated measure are completely determined from these coefficients. A very nice and short constructive proof of this last statement can be found in [10]. For more details on the classical theory of orthogonal polynomials on the unit circle we refer to [18–20].

Recently, it was shown in [7] that given a pair of real sequences $\{\{c_n\}_{n=1}^{\infty}, \{d_n\}_{n=1}^{\infty}\}$, with $\{d_n\}_{n=1}^{\infty}$ a positive chain sequence, we can associate a unique nontrivial probability measure μ on the unit circle. The associated measure is such that the corresponding Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$ are given by the relation

$$\alpha_{n-1} = \overline{\rho}_{n-1} \left[\frac{1 - 2m_n - ic_n}{1 - ic_n} \right], \quad n \geq 1,$$

where $\rho_0 = 1$, $\rho_n = \prod_{k=1}^{n} (1 - ic_k)/(1 + ic_k)$, $n \geq 1$ and $\{m_n\}_{n=0}^{\infty}$ is the minimal parameter sequence of $\{d_n\}_{n=1}^{\infty}$. In Theorem 2.1 we have given the complete information regarding this statement and its reciprocal.

Thus, we can look at the measure μ from the sequence of its associated Verblunsky coefficients or, equivalently, from the real sequences $\{c_n\}_{n=1}^{\infty}$ and $\{m_n\}_{n=1}^{\infty}$, where $\{m_n\}_{n=0}^{\infty}$ is the minimal parameter sequence of the chain sequence $\{d_n\}_{n=1}^{\infty}$. From the theory of chain sequences (see [6]) we have

$$d_n = (1 - m_{n-1}) m_n, \quad n \geq 1, \tag{1.1}$$

with $0 < m_n < 1$, $n \geq 1$ and $m_0 = 0$.

There are many results on classical OPUC theory regarding nontrivial probability measures with associated p-periodic sequence of Verblunsky coefficients. In the first section we summarize some of them. We denote by V_p the space of all nontrivial probability measures with associated p-periodic sequence of Verblunsky coefficients, i.e.,

$$\mu \in V_p \quad \text{if and only if} \quad \alpha_{n+p} = \alpha_n, \quad n \geq 0.$$

An important fact is that we can have a measure with periodic Verblunsky coefficients and such that the sequences of the associated pair $\{\{c_n\}_{n=1}^{\infty}, \{m_n\}_{n=1}^{\infty}\}$ are not periodic (see, for example, [8]). For this reason,