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Inverse spectral analysis for a class of infinite band symmetric matrices ☆

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ABSTRACT

This work deals with the direct and inverse spectral analysis for a class of infinite band symmetric matrices. This class corresponds to operators arising from difference equations with usual and *inner* boundary conditions. We give a characterization of the spectral functions for the operators and provide necessary and sufficient conditions for a matrix-valued function to be a spectral function of the operators. Additionally, we give an algorithm for recovering the matrix from the spectral function. The approach to the inverse problem is based on the rational interpolation theory.

1. Introduction

In this paper, the direct and inverse spectral analysis of a class of infinite real symmetric band matrices, denoted $\mathcal{M}(n,\infty)$, is carried out with emphasis in the inverse problems of characterization and reconstruction. The matrices under consideration, defined in the paragraphs below, arise from difference equations with initial and left endpoint boundary conditions together with the so called *inner* boundary conditions. Inner boundary conditions are given by degenerations of the diagonals (see the paragraphs above Definition 1 and equation (2.4)). Each matrix in $\mathcal{M}(n,\infty)$ generates uniquely a closed symmetric operator for which we give a spectral characterization. More specifically, we provide necessary and sufficient conditions for a matrix-valued function to be a spectral function of the operators stemming from our class of matrices (see Definition 5 and Theorems 5.1 and 5.2). As a byproduct of the spectral analysis of the operators

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corresponding to matrices in $\mathcal{M}(n,\infty)$, we find an if-and-only-if criterion for degeneration in terms of the properties of polynomials in an L_2 space (see Theorem 3.1).

Although the inverse spectral problems for Jacobi matrices have been studied extensively (see for instance [7–9,14,19,22–24,34–36] for the finite case and [10,11,13,14,20,21,37,38] for the infinite case), works dealing with band matrices, not necessarily tridiagonal, are not so abundant (see [5,17,18,27–29,32,41,42] for the finite case and [3,16] for the infinite case).

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and fix an orthonormal basis $\{\delta_k\}_{k=1}^\infty$ in it. We study the symmetric operator A whose matrix representation with respect to $\{\delta_k\}_{k=1}^\infty$ is a real symmetric band matrix which is denoted by $\mathcal{A}$ (see [2, Sec. 47] for the definition of the matrix representation of an unbounded symmetric operator).

We assume that the matrix $\mathcal{A}$ has $2n+1$ band diagonals ($n \in \mathbb{N}$), that is, $2n+1$ diagonals not necessarily zero. The band diagonals satisfy the following conditions. The band diagonal farthest from the main one, which is given by the diagonal matrix $\operatorname{diag}\{d_k^{(n)}\}_{k=1}^\infty$, denoted by $\mathcal{D}_n$, is such that, for some $m_1 \in \mathbb{N}$, all the numbers $d_1^{(n)}, \dots, d_{m_1-1}^{(n)}$ are positive and $d_k^{(n)} = 0$ for all $k \geq m_1$ with

$$m_1 > 1\,. \tag{1.1}$$

It may happen that all the elements of the sequence $\{d_k^{(n)}\}_{k\in\mathbb{N}}$ are positive which we convene to mean that $m_1 = \infty$.

Now, if $m_1 < \infty$, then the elements $\{d_{m_1+k}^{(n-1)}\}_{k=1}^\infty$ of the diagonal next to the farthest, $\mathcal{D}_{n-1}$, behave in the same way as the elements of $\mathcal{D}_n$, that is, there is m_2, satisfying

$$m_1 < m_2\,, \tag{1.2}$$

such that $d_{m_1+1}^{(n-1)}, \dots, d_{m_2-1}^{(n-1)} > 0$ and $d_k^{(n-1)} = 0$ for all $k \geq m_2$. Here, it is also possible that $m_2 = \infty$ in which case $d_k^{(n-1)} > 0$ for all $k > m_1$.

We continue applying the same rule as long as $m_1, \dots, m_j$ are finite. Thus, if $m_j < \infty$, there is m_{j+1}, satisfying

$$m_j < m_{j+1}\,, \tag{1.3}$$

such that $d_{m_j+1}^{(n-j)}, \dots, d_{m_{j+1}-1}^{(n-j)} > 0$ (here we assume that $m_j + 1 < m_{j+1}$) and $d_k^{(n-j)} = 0$ for all $k \geq m_{j+1}$. If $m_j = \infty$, then $d_k^{(n-j)} > 0$ for all $k > m_j$. Eventually, there is $j_0 \leq n-1$ such that $m_{j_0+1} = \infty$. We allow j_0 to be zero, which accordingly means that $m_1 = \infty$.

If $j_0 \geq 1$, as long as $j < j_0$, we say that the diagonal corresponding to $\mathcal{D}_{n-j}$ undergoes degeneration at m_{j+1}. Note that the diagonal corresponding to $\mathcal{D}_{n-j_0}$ does not degenerate. Also, j_0 defines the number of degenerations that the matrix $\mathcal{A}$ has.

Definition 1. For a natural number n, the set of matrices satisfying the above properties is denoted by $\mathcal{M}(n,\infty)$. The set of numbers $\{m_j\}_{j=1}^{j_0}$ characterizes the degenerations of the diagonals. For a matrix without degenerations, this set is empty.

A matrix in $\mathcal{M}(n,\infty)$ has the particular structure illustrated in Fig. 1. Due to transformations similar to the one given in [39, Lem. 1.6], this class of matrices is wider than it seems. We shall see in Section 2 that the rows where there are degenerations and the ones where there are not (cf. (2.12) and (2.10)) give rise to difference equations playing different roles in the spectral analysis of the operator corresponding to the matrix. This is so even when all entries denoted by gray squares are zero.