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Simplicity of eigenvalues and non-vanishing of eigenfunctions of a quantum graph

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ABSTRACT

We prove that after an arbitrarily small adjustment of edge lengths, the spectrum of a compact quantum graph with δ-type vertex conditions can be simple. We also show that the eigenfunctions, with the exception of those living entirely on a looping edge, can be made to be non-vanishing on all vertices of the graph. As an application of the above result, we establish that the secular manifold (also called "determinant manifold") of a large family of graphs has exactly two smooth connected components.

1. Introduction

A quantum graph is a metric graph equipped with a self-adjoint differential operator (usually of Schrödinger type) defined on the edges and matching conditions specified at the vertices. Every edge of the graph has a length assigned to it.

One of the fundamental questions of the spectral theory is that of presence in the spectrum of degenerate (or repeated) eigenvalues. In particular, it is usually the case that within a rich enough set of problems, the problems *with* degenerate eigenvalues form a small subset. In other words, unless a system has symmetries (which usually force degeneracy in the spectrum, see, for example, [20]), it is highly unlikely to have degenerate eigenvalues.

Mathematically, a classical result by Uhlenbeck [18] (see also [19] for a generalization) establishes generic simplicity of eigenvalues of the Laplace–Beltrami operator on compact manifolds, with respect to the set of all possible metrics on the manifold. Some generic properties of eigenfunctions are also established. Since then, various extensions and generalizations of this result have been proven for different circumstances (see, for example, [14] and references therein).

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On graphs, the question of simplicity of eigenvalues was considered by Friedlander in [13], who proved that the eigenvalues are generically simple with respect to the perturbation of the edge lengths of the graph. The proof is based on perturbation theory and applies to graphs with Neumann–Kirchhoff (NK) conditions only (see Section 2 for the definitions). When this article was in preparation, an outline of a shorter proof, under the same conditions, was released by Colin de Verdière [11].

In this work we consider a wider range of vertex conditions, namely the δ-type conditions on vertices of the graph. Furthermore, we also investigate the eigenfunctions, showing that generically they do not vanish on vertices, unless this is unavoidable due to presence of looping edges. Both of these results are important in applications, in particular all recent results on the number of zeros of graph eigenfunctions assume both the simplicity of eigenvalues and non-vanishing of eigenfunctions on vertices as a precondition (see [6,4,2,1,10] and references therein).

In the proof, the simplicity of eigenvalues and non-vanishing of eigenfunctions are tightly interconnected; each property is assisting in the proof of the other (the proof is done by induction). The proof is geometric in nature and uses local modifications of the graph to reduce it to previously considered case. In Section 6 of the paper we also consider an application of the result to the study of the secular manifold of a graph, showing that for large classes of graphs, the set of smooth points of the manifold has exactly two connected components.

We remark that from the general consideration one can deduce the result for generic choices of the vertex conditions. The challenge is to obtain it for a fixed choice of vertex conditions (and a generic choice of edge lengths). The existing proofs cannot be readily re-used for this purpose. While the original proof due to Friedlander [13] is very technical, the simpler proof by Colin de Verdière [11] relies on the properties of the so-called “secular manifold” for quantum graphs which does not exist for general δ-type conditions. Finally, we mention a result of Exner and Jex, where the simplicity of the ground state eigenvalue and positivity of the corresponding eigenfunction was established for graph with non-repulsive δ-type conditions [12].

2. Quantum graph Hamiltonian

We start by defining the quantum graph, following the notational conventions of [9]. Let $\Gamma = (V, E)$ be a connected metric graph with a set of vertices $V = \{v_j\}$ and edges $E = \{e_j\}$. Both sets V and E are assumed to be finite and the edges are of bounded length. We allow multiple edges between a given pair of vertices and the edges that loop from a vertex to itself (see also Remark 2.1 below).

A function f on Γ is a collection of functions $f_e(x)$ defined on each edge e. Consider the Laplace operator H defined by

$$H : f \mapsto -\frac{d^2 f}{dx^2},$$

acting on the functions that belong to the Sobolev $H^2(e)$ space on each edge e and satisfy the δ-type boundary conditions with coefficients α_v at the vertices of the graph,

$$\begin{cases} f(x) \text{ is continuous at } v \\ \sum_{e\in E_v} \dfrac{df}{dx_e}(v) = \alpha_v f(v), \end{cases} \tag{2.1}$$

where for each vertex v, the corresponding vertex condition α_v is a fixed real number. The set E_v is the set of edges joined at the vertex v; by convention, each derivative at a vertex is taken into the corresponding edge. We will often encounter the special case with $\alpha_v = 0$, which is known as the Neumann–Kirchhoff (NK) condition. The special value $\alpha_v = \infty$ should be taken to mean the Dirichlet condition $f(v) = 0$. Such condition will only be allowed at vertices of degree 1, as it effectively disconnects the edges if imposed at a vertex of degree 2 or higher. Conditions with $\alpha_v \neq 0, \infty$ will be called *Robin-type*.