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Anticommutation in the presentations of theta-deformed spheres

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ABSTRACT

We consider an analogue of the θ-deformed even spheres, modifying the relations demanded of the self-adjoint generator x in the usual presentation. In this analogue, x is given anticommutation relations with all of the other generators, as opposed to being central. The main result shows that these algebras satisfy a Borsuk–Ulam property that is visible in even K-theory.

1. Introduction

When a Rieffel deformation procedure [14] is applied to the function algebra $C(\mathbb{S}^k)$ of a sphere $\mathbb{S}^k$, the resulting C^*-algebra admits a succinct presentation [8,9,4].

Definition 1.1. An $n \times n$ matrix ρ is a *parameter matrix* if ρ has 1 in each diagonal entry, each entry of ρ is unimodular, and $\rho_{jk} = \overline{\rho_{kj}}$ for each j and k. Equivalently, $\rho_{jk} = e^{2\pi i \theta_{jk}}$ for a (nonunique) real, antisymmetric matrix θ.

Definition 1.2. Let ρ be an $n \times n$ parameter matrix. Then $C(\mathbb{S}^{2n-1}_\rho)$ and $C(\mathbb{S}^{2n}_\rho)$ are given by the following C^*-presentations.

$$\begin{aligned} C(\mathbb{S}^{2n-1}_\rho) &\cong C^*(z_1, \ldots, z_n \;\mid\; z_j z_j^* = z_j^* z_j, \quad z_k z_j = \rho_{jk} z_j z_k, \quad z_1 z_1^* + \ldots + z_n z_n^* = 1) \\ C(\mathbb{S}^{2n}_\rho) &\cong C^*(z_1, \ldots, z_n, x \;\mid\; z_j z_j^* = z_j^* z_j, \quad x = x^*, \quad z_k z_j = \rho_{jk} z_j z_k, \quad x z_j = z_j x, \\ &\qquad\qquad z_1 z_1^* + \ldots + z_n z_n^* + x^2 = 1) \end{aligned}$$

The presentations and notation illustrate the mentality suggested by the Gelfand–Naimark theorem, in that we view $C(\mathbb{S}^k_\rho)$ as a (noncommutative) function algebra with coordinate functions labeled z_j or x.

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The noncommutativity relations of $C(\mathbb{S}^k_\rho)$ vary continuously as a function of ρ, and $C(\mathbb{S}^{2n}_\rho)$ is realized as a quotient $C(\mathbb{S}^{2n+1}_\omega)/\langle z_{n+1} - z^*_{n+1}\rangle$, where ω is such that z_{n+1} is central and ρ is the upper left $n \times n$ submatrix of ω. However, this type of quotient is reasonable (that is, nondegenerate) even when z_{n+1} anticommutes with some of the other z_j. We consider below the quotients $C(\mathbb{S}^{2n+1}_\omega)/\langle z_{n+1} - z^*_{n+1}\rangle$ when anticommutation always occurs. As such, the presentation of these algebras is identical to that of $C(\mathbb{S}^{2n}_\rho)$, except anticommutation of x and z_j replaces commutation.

Definition 1.3. Let ρ be an $n \times n$ parameter matrix. Then $\mathfrak{R}^{2n}_\rho$ is defined by the following C^*-presentation.

$$\begin{aligned}\mathfrak{R}^{2n}_\rho \cong C^*(z_1, \ldots, z_n, x \mid\ & z_j z_j^* = z_j^* z_j, \quad x = x^*, \quad z_k z_j = \rho_{jk} z_j z_k, \quad x z_j = -z_j x, \\ & z_1 z_1^* + \ldots + z_n z_n^* + x^2 = 1)\end{aligned} \tag{1.4}$$

We adopt the convention that $\mathfrak{R}^{2n}$ without a subscript denotes $\mathfrak{R}^{2n}_\rho$ for ρ a matrix of all ones. In this algebra, $z_1, \ldots, z_n$ commute with each other and anticommute with x.

Each $\mathfrak{R}^{2n}_\rho$ may be realized as a noncommutative unreduced suspension (in the sense of [11], Definition 3.4) of the unital C^*-algebra $C(\mathbb{S}^{2n-1}_\rho)$ given by the antipodal map, which is the order two homomorphism that negates each generator of the standard presentation. In general, if β generates a $\mathbb{Z}_2$ action on a unital C^*-algebra A, then

$$\Sigma^\beta A := \{f \in C([0,1], A \rtimes_\beta \mathbb{Z}_2) : f(0) \in A, f(1) \in C^*(\mathbb{Z}_2)\}$$

defines the noncommutative unreduced suspension of $(A, \beta, \mathbb{Z}_2)$. When β is the trivial action, $\Sigma^\beta A$ is isomorphic to the unreduced suspension ΣA; this was considered in [5] and [11] in pursuit of noncommutative Borsuk–Ulam theory.

Theorem 1.5 *(Borsuk–Ulam). No continuous, odd maps exist from $\mathbb{S}^k$ to $\mathbb{S}^{k-1}$, and every continuous, odd map on $\mathbb{S}^{k-1}$ is homotopically nontrivial.*

In this context, a function f is odd if $f(-x) = -f(x)$ for all inputs x. Moreover, a function $f = f_0$ is homotopically trivial if there exists a homotopy f_t connecting f_0 to a constant function f_1. For example, the identity map on $\mathbb{S}^n$ is homotopically nontrivial, as $\mathbb{S}^n$ is not contractible. Noncommutative Borsuk–Ulam theorems [16,18,2,10,5,11,3] arise when a C^*-algebraic theorem generalizes the result of passing the traditional Borsuk–Ulam theorem, or one of its various restatements and extensions, through Gelfand–Naimark duality. In short, these theorems describe when an equivariant map between two related C^*-algebras cannot exist, or when an equivariant self-map must behave nontrivially with respect to some invariant. The θ-deformed odd spheres were considered for this purpose in [10] with $\mathbb{Z}_k$ rotation actions; the $\mathbb{Z}_2$ case is repeated below.

Theorem 1.6. *Let α denote the antipodal $\mathbb{Z}_2$-action on any θ-deformed sphere, which negates each generator of the standard presentation. If ρ and ω are $n \times n$ parameter matrices, then any α-equivariant, unital $*$-homomorphism from $C(\mathbb{S}^{2n-1}_\rho)$ to $C(\mathbb{S}^{2n-1}_\omega)$ must induce a non-constant map on $K_1 \cong \mathbb{Z}$.*

Corollary 1.7. *Fix $k \in \mathbb{Z}^+$ and parameter matrices γ and δ of the appropriate dimensions so that $C(\mathbb{S}^{k-1}_\gamma)$ and $C(\mathbb{S}^k_\delta)$ are defined. Then there are no antipodally equivariant, unital $*$-homomorphisms from $C(\mathbb{S}^{k-1}_\gamma)$ to $C(\mathbb{S}^k_\delta)$.*

The K-theory of θ-deformed spheres exactly matches that of the commutative case, as Rieffel deformation preserves K-theory [15], but there are more detailed descriptions [9,12].