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An upper bound for the norm of the Chebyshev polynomial on two intervals

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ABSTRACT

Let $E := [-1,\alpha] \cup [\beta,1]$, $-1 < \alpha < \beta < 1$, be the union of two real intervals and consider the Chebyshev polynomial of degree n on E, that is, that monic polynomial which is minimal with respect to the supremum norm on E. For its norm, called the n-th Chebyshev number of E, an upper bound in terms of elementary functions of α and β is given. The proof is based on results of N.I. Achieser in the 1930s in which the norm is estimated with the help of Zolotarev's transformation using Jacobi's elliptic and theta functions.

1. Introduction and main result

For $n \in \mathbb{N}$, let $\mathbb{P}_n$ (resp. $\hat{\mathbb{P}}_n$) denote the set of all polynomials (resp. monic polynomials) of degree n with complex coefficients. For a given set of two intervals

$$E := [-1,\alpha] \cup [\beta,1], \qquad -1 < \alpha < \beta < 1, \tag{1}$$

consider the Chebyshev polynomial of degree n on E, that is, that unique monic polynomial $T_n(x) = x^n + \ldots \in \hat{\mathbb{P}}_n$, which is minimal with respect to the supremum norm:

$$t_n(E) := \max_{x\in E} \left|T_n(x)\right| = \min_{P_n\in\hat{\mathbb{P}}_n} \max_{x\in E} |P_n(x)|. \tag{2}$$

The term $t_n(E)$ is called the n-th *Chebyshev number* of E. It is well known that the limit

$$\operatorname{cap}(E) := \lim_{n\to\infty} \sqrt[n]{t_n(E)} \tag{3}$$

exists, where $\operatorname{cap}(E)$ is called the *logarithmic capacity* (or *Chebyshev constant* or *transfinite diameter*) of E.

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The lower bound

$$t_n(E) \geq 2\,(\mathrm{cap}(E))^n \tag{4}$$

was derived by the author in [14], where in fact (4) was proved for all compact real sets E and the factor 2 is best possible. Concerning an upper bound for $t_n(E)$, the situation is more complicated. It was proved by Vilmos Totik [17], see also Widom [21], that there exists a constant C depending only on E such that for each $n \in \mathbb{N}$

$$t_n(E) \leq C(\mathrm{cap}(E))^n, \tag{5}$$

where in fact the above inequality was proved for a finite union of real intervals. For further results in this direction, see [18–20] and [7].

In this paper, based on the work of N.I. Achieser, see [3,2], and [1], we explicitly determine such a bound C in terms of simple functions of α and β. We will prove the following theorem.

Theorem 1. *Let $E := [-1, \alpha] \cup [\beta, 1]$ with $-1 < \alpha < \beta < 1$, then, for each $n \in \mathbb{N}$, the inequality*

$$t_n(E) \leq 2B(\mathrm{cap}(E))^n \tag{6}$$

holds with

$$B := \frac{2\sqrt{(1-\alpha)(1+\beta)} + \sqrt{(1+\alpha)(1-\beta)}}{\sqrt[4]{(1-\alpha^2)(1-\beta^2)}}. \tag{7}$$

Based on a result of Haliste [8], Dubinin and Karp [6] obtained a very precise upper bound for the logarithmic capacity of several intervals, see also [16, Theorem 6] for a discussion for the two intervals case. Using their inequality and inequality (6) with (7), we get the following upper bound for the Chebyshev number $t_n(E)$:

$$\begin{aligned} t_n(E) \leq\ & \frac{4\sqrt{(1-\alpha)(1+\beta)} + 2\sqrt{(1+\alpha)(1-\beta)}}{\sqrt[4]{(1-\alpha^2)(1-\beta^2)}} \\ & \times \frac{1}{4^n}\Big(\sqrt{(1+\alpha)(1+\beta)} + \sqrt{(1-\alpha)(1-\beta)}\Big)^n \end{aligned} \tag{8}$$

The paper is organized as follows. In Section 2, we introduce Jacobi's elliptic and theta functions and collect many useful identities, which are necessary for the proofs. In Section 3, we recall (with detailed proofs) the results of Achieser and prove our main result. Sections 4 and 5 are devoted, on the one hand, to the special case of two symmetric intervals, where we slightly improve Achieser bounds for $t_n(E)$, and, on the other hand, to the discussion of Achieser's famous result on the asymptotic behavior of the sequence

$$W_n(E) := \frac{t_n(E)}{\big(\mathrm{cap}(E)\big)^n}. \tag{9}$$

The notation $W_n(E)$ (for general compact sets E in the complex plane) was introduced in [7] in honor of Harold Widom and his seminal paper [21]. In the last section, we discuss the very recent upper bound given by Christiansen, Simon and Zinchenko [5] for the two interval case.