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Pure infiniteness and ideal structure of C^*-algebras associated to Fell bundles

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ABSTRACT

We investigate structural properties of the reduced cross-sectional algebra $C_r^*(\mathcal{B})$ of a Fell bundle $\mathcal{B}$ over a discrete group G. Conditions allowing one to determine the ideal structure of $C_r^*(\mathcal{B})$ are studied. Notions of aperiodicity, paradoxicality and $\mathcal{B}$-infiniteness for the Fell bundle $\mathcal{B}$ are introduced and investigated by themselves and in relation to the partial dynamical system dual to $\mathcal{B}$. Several criteria of pure infiniteness of $C_r^*(\mathcal{B})$ are given. It is shown that they generalize and unify corresponding results obtained in the context of crossed products, by the following duos: Laca, Spielberg [34]; Jolissaint, Robertson [21]; Sierakowski, Rørdam [47]; Giordano, Sierakowski [18] and Ortega, Pardo [39]. For exact, separable Fell bundles satisfying the residual intersection property primitive ideal space of $C_r^*(\mathcal{B})$ is determined. The results of the paper are shown to be optimal when applied to graph C^*-algebras. Applications to a class of Exel–Larsen crossed products are presented.

1. Introduction

Many of C^*-algebras studied in literature are equipped with a natural additional structure which can be used to study their properties. This structure can be exhibited by a group co-action (or a group action if the underlying group is abelian) or more generally by a group grading of the C^*-algebra. It allows one to investigate the C^*-algebra by means of the associated Fell bundle of subspaces determining the grading. Fell bundles over discrete groups proved to be a convenient framework for studying crossed products corresponding to global or partial group actions, and were successfully applied to diverse classes of C^*-algebras, [16,18,1]. Moreover, the approach based on Fell bundles has recently gained an increased interest in an analysis of C^*-algebras associated to generalized graphs [7], Nica–Pimsner algebras [8], and Cuntz–Pimsner algebras [33,2] associated to product systems over semigroups. We remark that, in contrast to most of appli-

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cations in [16,18,1], in the latter case the core C^*-algebra corresponding to the unit in the group, as a rule, is non-commutative. The present paper is devoted to investigations of the ideal structure, pure infiniteness and related features of the reduced cross-sectional algebras $C^*_r(\mathcal{B})$ arising from a Fell bundle $\mathcal{B} = \{B_t\}_{t\in G}$ over a discrete group G with the unit fiber B_e being genuinely a non-commutative C^*-algebra. One of our primary aims is to give convenient C^*-dynamical conditions on $\mathcal{B}$ that lead to a coherent treatment unifying various approaches to pure infiniteness of crossed products by group actions [34,21,47,18], and that are applicable to C^*-algebras arising from semigroup structures. Actually, for a class of Fell bundles we consider, the C^*-algebra $C^*_r(\mathcal{B})$ has the ideal property, and it is known that in the presence of this property pure infiniteness [23, Definition 4.1] is equivalent to strong pure infiniteness [24, Definition 5.1]. Additionally, if $C^*_r(\mathcal{B})$ is separable we provide a description of the primitive spectrum of $C^*_r(\mathcal{B})$. This together with known criteria for nuclearity of $C^*_r(\mathcal{B})$, cf. [16, Proposition 25.10], forms a full toolkit for producing and analyzing graded C^*-algebras that undergo Kirchberg's classification (up to stable isomorphism) via ideal system equivariant KK-theory [22].

In order to detect pure infiniteness of a non-simple C^*-algebra, one needs to understand its ideal structure. The general algebraic necessary and sufficient conditions assuring that the ideals in the ambient algebra are uniquely determined by their intersection with the core are known. These conditions are exactness and the residual intersection property. They were introduced in the context of crossed products in [48], then generalized to partial crossed products in [18] and to cross-sectional algebras in [1]. We give a metric characterization of the intersection property using a notion of topological grading, and we shed light on the notion of exactness of a Fell bundle $\mathcal{B} = \{B_t\}_{t\in G}$ by characterizing it in terms of graded and Fourier ideals in $C^*_r(\mathcal{B})$.

An important dynamical condition implying the (residual) intersection property of $\mathcal{B}$ is (residual) topological freeness of a dual partial dynamical system $(\{\widehat{D}_t\}_{t\in G}, \{\widehat{h}_t\}_{t\in G})$ defined on the spectrum $\widehat{B}_e$ of the core B_e. This result is well-known for crossed products, cf. [4]. Recently, it was generalized to cross-sectional algebras of saturated Fell bundles by the authors of the present paper [33], and to general Fell bundles by Beatriz Abadie and Fernando Abadie [1]. The system $(\{\widehat{D}_t\}_{t\in G}, \{\widehat{h}_t\}_{t\in G})$ is very useful in investigation of the ideal structure of $C^*_r(\mathcal{B})$. In particular, it factorizes to a partial dynamical system on the primitive spectrum $\mathrm{Prim}(B_e)$ of B_e, and we show that for exact, separable Fell bundles satisfying the residual intersection property the primitive ideal space of $C^*_r(\mathcal{B})$ can be identified with the quasi-orbit space of this dual action on $\mathrm{Prim}(B_e)$. We show below that this result applied to graph C^*-algebras $C^*(E)$ with their natural $\mathbb{Z}$-gradings gives a new way of determining primitive ideal space of $C^*(E)$ for an arbitrary graph E satisfying Condition (K). The latter description was originally obtained in [5] by different methods.

In general, the aforementioned dual system is not well suited for determining pure infiniteness of $C^*_r(\mathcal{B})$, as it gives no control on positive elements. Therefore we introduce a concept of aperiodicity for Fell bundles, which is related to the aperiodicity condition for C^*-correspondences introduced by Muhly and Solel in [37]. One should note that the origins of this notion go back to the work of Kishimoto [25] and Olesen and Pedersen [38] where the close relationship between this condition and properties of the Connes spectrum was revealed. More recently, similar aperiodicity conditions were investigated in the context of partial actions by Giordano and Sierakowski in [18]. The precise relationship between aperiodicity and topological freeness is not clear, however we prove that, under the additional hypothesis that the primitive ideal space of B_e is Hausdorff, topological freeness of the partial dynamical system on $\mathrm{Prim}(B_e)$ implies aperiodicity of $\mathcal{B}$. We show that a Fell bundle associated to a graph E is aperiodic if and only if E satisfies Condition (L).

Exploiting ideas of Rørdam and Sierakowski [47], modulo observations made in [30], we prove that if a Fell bundle $\mathcal{B}$ is exact, residually aperiodic, and B_e has the ideal property or contains finitely many $\mathcal{B}$-invariant ideals,[1] then $C^*_r(\mathcal{B})$ has the ideal property and pure infiniteness of $C^*_r(\mathcal{B})$ is equivalent to proper infiniteness of every non-zero positive element in B_e (treated as an element in $C^*_r(\mathcal{B})$). If additionally B_e

[1] In the initially submitted manuscript we considered only the case when B_e has the ideal property.