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Fundamental solutions for the two dimensional affine group and $\mathbb{H}^{n+1}$

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ABSTRACT

We derive the wave and heat kernels on the $ax+b$ group, as well as the fundamental solution of the group Laplacian. We make particular use of the Kontorovich–Lebedev transform and a recent result of the author to produce new expressions for these kernels. Our results easily extend to the hyperbolic space $\mathbb{H}^{n+1}$ for any n and the explicit formulas are given in n dimensions.

1. Introduction

The group of affine transformation of $\mathbb{R}$, which is also known as the $ax+b$ group, is the set of matrices

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}.$$

G is the simplest solvable Lie group. The isomorphism

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a^{1/2} & a^{-1/2}b \\ 0 & a^{-1/2} \end{pmatrix},$$

embeds this as a subgroup of $SL(2,\mathbb{R})$. G has an action on the line defined by

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ 1 \end{pmatrix}, \tag{1.1}$$

whence the name. The group generated by the vector fields $X = y\frac{\partial}{\partial x}$ and $Y = y\frac{\partial}{\partial y}$ is easily seen to be isomorphic to G. In this paper we are interested in the Laplacian on G, which we write as

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$$\mathcal{L} = X^2 + Y^2 = y^2 \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right) + y \frac{\partial}{\partial y}. \tag{1.2}$$

From this, we may study three natural PDEs on G. Specifically:

(i) *the heat equation* $u_t = \mathcal{L}u$
(ii) *the wave equation* $u_{tt} = \mathcal{L}u$
(iii) *Laplace's equation* $\mathcal{L}u = 0$.

In this paper we solve the appropriate initial and boundary value problems for the heat and wave equations. Fundamental solutions of these operators have been obtained using various means, often more from a geometric point of view. We will not attempt to detail the literature, but See Davies' book [6] for a starting point. The aim of this paper is to derive fundamental solutions of these classical operators using integral transform theory. Specifically the Fourier and Kontorovich–Lebedev (KL) transforms. We also use a recent result of the author. We will produce what appear to be new expressions for the fundamental solutions of the heat and wave equations and establish an interesting new relationship between the heat and wave kernels. Specifically we show how the heat kernel can be derived in terms of the wave kernel, avoiding the usual functional calculus approach to establishing such relations We then give an expression for the fundamental solution of (iii) also in terms of the wave kernel. Higher dimensional extensions are discussed in the final section.

2. The wave equation

Our aim is to solve the Cauchy problem

$$\begin{aligned} u_{tt} &= y^2(u_{yy} + u_{xx}) + yu_y,\ x \in \mathbb{R},\ y \geq 0,\ t > 0 \\ u(x, y, 0) &= f(x, y) \\ u_t(x, y, 0) &= g(x, y). \end{aligned}$$

We require g and f to be defined on the whole of $\mathbb{R}^2$ and

$$\int_0^\infty \int_{-\infty}^\infty \frac{1}{y} |g(x, y)| dx dy < \infty, \tag{2.1}$$

and

$$\int_0^\infty \int_{-\infty}^\infty \frac{1}{y} |\widehat{g}(x, y)| dx dy < \infty. \tag{2.2}$$

f must satisfy the same inequalities. Here and throughout,

$$\widehat{g}(\xi, y) = \int_{-\infty}^\infty g(x, y) e^{-ix\xi} dx, \tag{2.3}$$

the Fourier transform in the first variable. We will base our analysis on the Kontorovich–Lebedev transform and so we further assume that