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Turán type inequalities for Struve functions [☆]

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ABSTRACT

Some Turán type inequalities for Struve functions of the first kind are deduced by using various methods developed in the case of Bessel functions of the first and second kind. New formulas, like Mittag–Leffler expansion, infinite product representation for Struve functions of the first kind, are obtained, which may be of independent interest. Moreover, some complete monotonicity results and functional inequalities are deduced for Struve functions of the second kind. These results complement naturally the known results for a particular case of Lommel functions of the first kind, and for modified Struve functions of the first and second kind.

1. Turán type inequalities for Struve functions of the first kind

Let us start with a well-known relation between Bessel functions of the first kind J_ν and Struve functions of the first kind $\mathbf{H}_\nu$. Namely, for all $n \in \{0, 1, \dots\}$ and $x \in \mathbb{R}$ we have [11, p. 291]

$$\mathbf{H}_{-n-\frac{1}{2}}(x) = (-1)^n J_{n+\frac{1}{2}}(x).$$

Now, let us recall the Turán type inequality for Bessel functions of the first kind, that is,

$$J_\nu^2(x) - J_{\nu-1}(x)J_{\nu+1}(x) \geq 0, \tag{1.1}$$

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where $x \in \mathbb{R}$ and $\nu > -1$. Combining the above relation with (1.1) we obtain the Turán type inequality

$$\mathbf{H}_{\nu}^2(x) - \mathbf{H}_{\nu-1}(x)\mathbf{H}_{\nu+1}(x) \geq 0, \tag{1.2}$$

which holds for all $x \in \mathbb{R}$ and $\nu \in \left\{-\frac{1}{2}, -\frac{3}{2}, \dots\right\}$. Moreover, since for $x \in \mathbb{R}$ and $\nu \geq 0$ the Turán type inequality (1.1) can be improved as

$$J_{\nu}^2(x) - J_{\nu-1}(x)J_{\nu+1}(x) \geq \frac{1}{\nu+1}J_{\nu}^2(x),$$

it follows that the inequality (1.2) can be improved too as

$$\mathbf{H}_{\nu}^2(x) - \mathbf{H}_{\nu-1}(x)\mathbf{H}_{\nu+1}(x) \geq \frac{1}{1-\nu}\mathbf{H}_{\nu}^2(x), \tag{1.3}$$

which holds for all $x \in \mathbb{R}$ and $\nu \in \left\{-\frac{1}{2}, -\frac{3}{2}, \dots\right\}$. For more details on the above Turán type inequalities for Bessel functions of the first kind we refer to the papers [12,14–16] and also to the survey paper [2]. Taking into account the above inequalities it is natural to ask whether the Turán type inequalities (1.2) and/or (1.3) hold true for other values of ν. In this paper we will concentrate on this problem and we present some interesting results concerning Turán type inequalities for Struve functions of the first and second kind. As we can see below the analysis of Struve functions is somewhat more complicated than that of Bessel functions, however, its nature is quite similar for some values of ν. This section is devoted to Turán type inequalities for Struve functions of the first kind, while the next section contains some results, like Turán type inequalities and complete monotonicity results on Struve functions of the second kind. Before we present the main results of this section we first show some preliminary results which will be used in the sequel and which may be of independent interest. Since Struve functions are frequently used in many places in physics and applied mathematics, we believe that our results may be useful for other scientists interested in Struve functions. We also note that the analogous results for modified Struve functions of the first and second kind were already deduced by Baricz and Pogány [3,4] by using the techniques developed in the case of modified Bessel functions of the first and second kind. Moreover, the results presented in this section complement naturally the known results for a particular case of Lommel functions of the first kind, obtained recently by Baricz and Koumandos [1].

The next result is analogous to the well-known result for Bessel functions of the first kind.

Lemma 1. *If $|\nu| \leq \frac{1}{2}$, then the Hadamard factorization of the real entire function $\mathcal{H}_{\nu} : \mathbb{R} \to (-\infty, 1]$, defined by $\mathcal{H}_{\nu}(x) = \sqrt{\pi}2^{\nu}x^{-\nu-1}\Gamma\left(\nu + \frac{3}{2}\right)\mathbf{H}_{\nu}(x)$, reads as follows*

$$\mathcal{H}_{\nu}(x) = \prod_{n\geq 1}\left(1 - \frac{x^2}{h_{\nu,n}^2}\right), \tag{1.4}$$

where $h_{\nu,n}$ stands for the nth positive zero of the Struve function $\mathbf{H}_{\nu}$. The above infinite product is absolutely convergent and if $|\nu| \leq \frac{1}{2}$ and $x \neq h_{\nu,n}$, $n \in \{1, 2, \dots\}$, then the Mittag–Leffler expansion of the Struve function $\mathbf{H}_{\nu}$ is as follows

$$\frac{\mathbf{H}_{\nu-1}(x)}{\mathbf{H}_{\nu}(x)} = \frac{2\nu+1}{x} + \sum_{n\geq 1}\frac{2x}{x^2 - h_{\nu,n}^2}. \tag{1.5}$$

Proof. By using the power series expansion of the Struve function $\mathbf{H}_{\nu}$ [11, p. 288]

$$\mathbf{H}_{\nu}(x) = \left(\frac{x}{2}\right)^{\nu+1}\sum_{n\geq 0}\frac{(-1)^n\left(\frac{x}{2}\right)^{2n}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\nu+\frac{3}{2}\right)}$$