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Geometric properties of surfaces with the same mean curvature in $\mathbb{R}^3$ and $\mathbb{L}^3$

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A B S T R A C T

Spacelike surfaces in the Lorentz–Minkowski space $\mathbb{L}^3$ can be endowed with two different Riemannian metrics, the metric inherited from $\mathbb{L}^3$ and the one induced by the Euclidean metric of $\mathbb{R}^3$. It is well known that the only surfaces with zero mean curvature with respect to both metrics are open pieces of the helicoid and of spacelike planes. We consider the general case of spacelike surfaces with the same mean curvature with respect to both metrics. One of our main results states that those surfaces have non-positive Gaussian curvature in $\mathbb{R}^3$. As an application of this result, jointly with a general argument on the existence of elliptic points, we present several geometric consequences for the surfaces we are considering. Finally, as any spacelike surface in $\mathbb{L}^3$ is locally a graph, our surfaces are locally determined by the solutions to the $H_R = H_L$ surface equation. Some uniqueness results for the Dirichlet problem associated to this equation are given.

1. Introduction

A hypersurface in the Lorentz–Minkowski space $\mathbb{L}^{n+1}$ is said to be spacelike if its induced metric is a Riemannian one. We can endow a spacelike hypersurface in $\mathbb{L}^{n+1}$ with another Riemannian metric, the one inherited from the Euclidean space $\mathbb{R}^{n+1}$. Therefore, we can consider two different mean curvature functions on a spacelike hypersurface, the mean curvature function related to the metric induced by $\mathbb{R}^{n+1}$, that we will denote by H_R, and the one related to the metric inherited from $\mathbb{L}^{n+1}$, H_L.

A hypersurface in $\mathbb{R}^{n+1}$ is said to be minimal if its mean curvature function vanishes identically, that is $H_R \equiv 0$. Analogously, a spacelike hypersurface in $\mathbb{L}^{n+1}$ is said to be maximal if $H_L \equiv 0$. This terminology comes from the fact that minimal (maximal) hypersurfaces locally minimize (maximize) area among all nearby hypersurfaces sharing the same boundary, see [14].

The study of minimal and maximal hypersurfaces is a topic of wide interest. One of the main results about the global geometry of minimal surfaces is the well-known Bernstein theorem, proved by Bernstein [5]

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in 1915, which states that the only entire minimal graphs in $\mathbb{R}^3$ are the planes. Some decades later, in 1970, Calabi [7] proved its analogous version for spacelike surfaces in the Lorentz–Minkowski space, the Calabi–Bernstein theorem, which states that the only entire maximal graphs in $\mathbb{L}^3$ are the spacelike planes. An important difference between both results is that the Bernstein theorem can be extended to minimal graphs in $\mathbb{R}^{n+1}$ up to dimension $n = 7$, but it is no longer true for bigger dimensions [6]. However, the Calabi–Bernstein theorem holds true for any dimension as it was proved by Calabi [7] for dimension $n \leq 4$, and by Cheng and Yau [8] for arbitrary dimension.

It is interesting to note that any complete spacelike hypersurface in $\mathbb{L}^{n+1}$ is necessarily an entire graph over any spacelike hyperplane, see [3, Proposition 3.3]. Therefore, the Calabi–Bernstein theorem can also be expressed in a parametric way by asserting that the only complete maximal hypersurfaces in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. This parametric version is not true in $\mathbb{R}^{n+1}$, indeed there exists a wide family of examples of non-trivial complete minimal hypersurfaces in $\mathbb{R}^{n+1}$.

As an immediate consequence of the above results, we conclude that the only complete hypersurfaces that are simultaneously minimal in $\mathbb{R}^{n+1}$ and maximal in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes.

Going a step further, we can consider spacelike hypersurfaces with the same constant mean curvature functions H_R and H_L. Heinz [12], Chern [9] and Flanders [10] proved that the only entire graphs with constant mean curvature H_R in $\mathbb{R}^{n+1}$ are the minimal graphs. There are examples of entire spacelike graphs with constant mean curvature H_L in $\mathbb{L}^{n+1}$ which are not maximal, for instance the hyperbolic spaces. However, taking into account the Calabi–Bernstein theorem, we conclude again that the only complete spacelike hypersurfaces in $\mathbb{L}^{n+1}$ with the same constant mean curvature functions H_R and H_L are the spacelike hyperplanes.

Without assuming any completeness hypothesis, Kobayashi [13] studied the problem for $H_R = H_L = 0$ in the 2-dimensional case. He showed that the only surfaces that are simultaneously minimal and maximal are open pieces of a spacelike plane or of a helicoid in the region where the helicoid is spacelike. However, nothing is known neither for bigger dimension nor for more general mean curvature functions H_R and H_L.

Our purpose is to study some local and global geometric properties of the spacelike surfaces in $\mathbb{L}^3$ such that $H_R = H_L$, not necessarily constant. Although we will focus on dimension 2, some results are still true in arbitrary dimension.

It is well known that any spacelike surface can be locally seen as a graph over an open domain of a spacelike plane, which without loss of generality can be supposed to be the plane $x_3 = 0$, see [14]. That is, a spacelike surface is locally defined by a smooth function u. Therefore, the functions H_R and H_L can be written in terms of such a function u and its partial derivatives. In this way, the identity $H_R = H_L$ becomes a quasilinear elliptic partial differential equation, everywhere except at those points at which the Euclidean gradient of u vanishes, where the equation is parabolic.

In Section 2 we present some basic preliminaries on spacelike hypersurfaces in $\mathbb{L}^{n+1}$ and their mean curvature functions with respect to the metrics inherited from $\mathbb{R}^{n+1}$ and $\mathbb{L}^{n+1}$. In the next section we state a result on the existence of an elliptic point in a hypersurface of $\mathbb{R}^{n+1}$ under some appropriate assumptions, and we see that the same result holds for spacelike hypersurfaces in $\mathbb{L}^{n+1}$. In Section 4 we consider spacelike surfaces in $\mathbb{L}^3$ such that $H_R = H_L$. We prove that for those surfaces K_R is always non-positive, and if the mean curvature does not vanish at a point, then the surface is locally non-convex at that point, **Theorem 4**. From this theorem, as well as from the result on the existence of an elliptic point, we get some consequences to which the rest of the section is devoted. Specifically, we prove the following theorems.

Theorem 5. *Let Σ be a compact spacelike surface with (necessarily) non-empty boundary such that $H_R = H_L$. Then Σ is contained in the convex hull of its boundary.*

Theorem 6. *The only spacelike graphs Σ_u in $\mathbb{L}^3$ defined over a domain $\Omega \subseteq \mathbb{R}^2$ of infinite width, with $H_R = H_L$, and asymptotic to a spacelike plane, are (pieces of) spacelike planes.*