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The existence of strong solutions to steady motion of electrorheological fluids in 3D cubic domain

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ABSTRACT

In this paper, we prove the existence of strong solutions to steady motion of electrorheological fluids without restriction on a smallness of external force in 3D cubic domain. And we improve the lower bound on $p(x)$ for regularity of solutions to the problem neglecting convective term even if $p(x)$ is a constant.

1. Introduction and main results

We are concerned with the following system

$$\begin{cases} -\nabla \cdot \left((1+|\mathcal{D}u|^2)^{\frac{p(x)-2}{2}} \mathcal{D}u \right) + (u \cdot \nabla)u + \nabla\pi = f, & \text{in } \Omega, \\ \nabla \cdot u = 0, & \text{in } \Omega \end{cases} \tag{1.1}$$

where u is the velocity, π the pressure, f the external force, $p(x)$ a prescribed function, and $\mathcal{D}u$ the symmetric part of the velocity gradient, i.e. $\mathcal{D}u := \frac{1}{2}(\nabla u + \nabla u^T)$. Here we consider a cubic domain $\Omega = (0,\ 1)^3$. We set $\Gamma := \{x \in \partial\Omega : |x_1|, |x_2| < 1, x_3 = 0\} \cup \{x \in \partial\Omega : |x_1|, |x_2| < 1, x_3 = 1\}$. The Dirichlet boundary condition will be imposed only on Γ. The problem will be assumed periodic on the other faces, i.e.

$$u|_\Gamma = 0, \qquad u \text{ is } x'\text{-periodic}, \tag{1.2}$$

where by x'-periodic we mean periodic of period 1 both in x_1 and x_2. This allows us to work in a bounded domain and simultaneously with a flat boundary.

This kind of systems models incompressible electrorheological fluids (ERF) with shear dependent viscosity, which are viscous fluids characterized by their ability to highly change in their mechanical properties

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when an electric field is applied (cf. [20,17]). In the last twenty years, the interest in the study of ERFs has increased. Since $p(x)$ isn't usually constant in system (1.1), the natural functional setting for studying (1.1) are variable exponent Lebesgue and Sobolev spaces (for its definition, see Section 2). For 3D bounded domains, in [20], the existence of local strong solutions to ERF's steady motions with Dirichlet boundary conditions has been obtained if $1.8 < p_1 := \inf p(x) \leq p_2 := \sup p(x) < 6$. In [15], Ettwein, Růžička showed the existence of $W^{2,\frac{3\hat{p}(x)}{\hat{p}(x)+1}}_{\text{loc}}$-solutions (where $\hat{p}(x) := \min\{p(x), 2\}$) without the artificial upper bound $p_2 < 6$. In [11], Crispo, Grisanti proved the existence and uniqueness of $C^{1,\gamma}(\overline{\Omega}) \cap W^{2,2}(\Omega)$-solution corresponding to small data, without further restrictions on the bounds p_1, p_2. In [1], Acerbi and Mingione proved interior partial regularity in any dimension if $p_1 > \frac{3n}{n+2}$. For 2D bounded domains, we refer to [9,12].

On the other hand, global regularity for solutions to the generalized stationary Navier–Stokes system of equations with shear dependent viscosity ($p \equiv constant$) has been studied in recent years [3,2,4–7]. In [3] it is shown $W^{2,\frac{3p(p-1)}{p^2-2p+3}}$-global regularity of solutions to (1.1), (1.2) for $\frac{15}{8} < p < 2$, in [7], $W^{2,\frac{4p-2}{p+1}}$-global regularity of ones for $\frac{7+\sqrt{35}}{7} < p < 2$, and in [2], the same regularity as in [7] for $\frac{20}{11} < p < 2$. In [4], it is shown the same results as in [2] in the case of Dirichlet condition on smooth boundary. In [6], the authors proved $W^{2,\frac{4(p+1)}{3p}}$-regularity of solutions in smooth domain for $p > 2$. In particular, it is shown global regularity of solutions to the generalized Stokes equations under the restriction $p > \frac{3}{2}$ [3,2,4–7]. For nonsteady cases, we refer to [8,14]. In [5], the author studied regular boundary points for non-homogeneous boundary value problems for the so called p-Laplacian operator. For the problems related to p-Laplacian operator, we refer to monograph [19].

We are interested in the existence of strong solutions to (1.1), (1.2). At first, we study regularity of solutions to problem neglecting convective term;

$$\begin{cases} -\nabla \cdot \left((1+|\mathcal{D}u|^2)^{\frac{p(x)-2}{2}}\mathcal{D}u\right) + \nabla\pi = f, & \text{in } \Omega, \\ \nabla \cdot u = 0, & \text{in } \Omega. \end{cases} \tag{1.3}$$

The regularity for solutions to the problem (1.3), (1.2) is the starting point for proving existence of strong solutions to the problem (1.1), (1.2). Actually, we assume that the convective term is a part of the right-hand side. A crucial point is to prove sharp estimates for the solutions u of (1.3), (1.2) in terms of the right-hand side f. For our purpose, we define (for definition of spaces $\mathbf{W}^{1,p(x)}(\Omega)$, $L^{q(x)}(\Omega)$, see Section 2)

$$\mathbf{V}_{p(x)} := \{u \in \mathbf{W}^{1,p(x)}(\Omega) \mid \nabla \cdot u = 0,\ u \text{ satisfies (1.2)}\},$$
$$L^{q(x)}_0(\Omega) := \{\, g \in L^{q(x)}(\Omega) | \int_\Omega g dx = 0\}.$$

Denote $\Omega_1 := \{x \in \Omega : p(x) < 2\}$, $\Omega_2 := \Omega \setminus \Omega_1$. Define $p_{1,1} := \text{essinf}_{x\in\Omega_1} m(x)$. By $p'(x)$ we denote the conjugate function of $p(x)$. The symbol $\nabla^2_* u$ denotes any of the second order derivatives except for the derivatives $\partial^2_{33}u_1$, $\partial^2_{33}u_2$. Similarly, $\nabla_* u$ may denote any first order partial derivative, except for $\partial_3 u$.

Definition 1.1. We say that function u is a weak solution of the problem (1.3), (1.2) if $u \in \mathbf{V}_{p(x)}$ and it satisfies

$$\int_\Omega (1+|\mathcal{D}u|^2)^{\frac{p(x)-2}{2}}\mathcal{D}u : \mathcal{D}\phi dx = \int_\Omega f \cdot \phi dx, \forall \phi \in \mathbf{V}_{p(x)}. \tag{1.4}$$

Note that the existence and uniqueness of the above solution is shown if $f \in \mathbf{L}^{p'(x)}(\Omega)$, and $1 \leq p(x) < \infty$ (see [20]).