J. Math. Anal. Appl. 445 (2017) 1047–1070

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Circular free spectrahedra

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ARTICLE INFO

Article history:
Received 2 May 2016
Available online 17 August 2016
Submitted by N. Young

Keywords:
Linear matrix inequality (LMI)
Spectrahedron
Matrix convex set
Free convexity
Circular domain
Invariant polynomial

ABSTRACT

This paper considers matrix convex sets invariant under several types of rotations. It is known that matrix convex sets that are free semialgebraic are solution sets of Linear Matrix Inequalities (LMIs); they are called free spectrahedra. We classify all free spectrahedra that are circular, that is, closed under multiplication by e^{it}: up to unitary equivalence, the coefficients of a minimal LMI defining a circular free spectrahedron have a common block decomposition in which the only nonzero blocks are on the superdiagonal. A matrix convex set is called free circular if it is closed under left multiplication by unitary matrices. As a consequence of a Hahn–Banach separation theorem for free circular matrix convex sets, we show the coefficients of a minimal LMI defining a free circular free spectrahedron have, up to unitary equivalence, a block decomposition as above with only two blocks. This paper also gives a classification of those noncommutative polynomials invariant under conjugating each coordinate by a different unitary matrix. Up to unitary equivalence such a polynomial must be a direct sum of univariate polynomials.
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1. Introduction

For square matrices A, B, write $A \preceq B$ (resp. $A \prec B$) to express that $B - A$ is positive semidefinite (resp. positive definite). Given a g-tuple $A = (A_1, \dots A_g) \in M_d(\mathbb{C})^g$, let $\Lambda_A(x)$ denote the linear matrix polynomial

$$\Lambda_A(x) = \sum_{j=1}^{g} A_j x_j \tag{1.1}$$

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[1] Research supported by the NSF grant DMS 1201498, and the Ford Motor Company.
[2] Supported by the Marsden Fund Council of the Royal Society of New Zealand. Partially supported by the Slovenian Research Agency grants P1-0222 and L1-6722.
[3] Research supported by the NSF grant DMS-1361501.

http://dx.doi.org/10.1016/j.jmaa.2016.07.011
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and let L_A denote the (**symmetric monic**) **linear pencil**

$$L_A(x) = I_d - \sum_{j=1}^{g} A_j x_j - \sum_{j=1}^{g} A_j^* x_j^* = I_d - \Lambda_A(x) - \Lambda_A(x)^*. \tag{1.2}$$

The **spectrahedron** $\mathscr{S}_A$ is the set of all $x \in \mathbb{C}^g$ satisfying the **linear matrix inequality** (**LMI**) $L_A(x) \succeq 0$. Spectrahedra and LMIs are ubiquitous in control theory [25,5] and optimization [4]. Indeed LMIs are at the heart of the subject called semidefinite programming.

This article investigates spectrahedra from the perspective of the emerging areas of free convexity [7,9,11,12,18,28–30] and free analysis [1,3,16,19,21,24,26,27]. In free analysis we are interested in matrix variables and evaluate a linear pencil on g-tuples $X = (X_1, \ldots, X_g) \in M_n(\mathbb{C})^g$ according to the formula

$$L(X) = I_d \otimes I_n - \sum_{j=1}^{g} A_j \otimes X_j - \sum_{j=1}^{g} A_j^* \otimes X_j^*. \tag{1.3}$$

For positive integers n, let

$$\mathcal{D}_A(n) = \big\{X \in M_n(\mathbb{C})^g : L_A(X) \succeq 0\big\}. \tag{1.4}$$

The sequence $\mathcal{D}_A = (\mathcal{D}_A(n))_n$ is called a **free spectrahedron**. It is the set of all solutions to the ampliated LMI corresponding to L_A. In particular, $\mathcal{D}_A(1) = \mathscr{S}_A$. Free spectrahedra are closely connected with operator systems for which [13,20,2] are a few recent references. In a different direction they provide a model for convexity phenomena in linear system engineering problems described entirely by signal flow diagrams [8].

The main results of this article characterize free spectrahedra and free polynomials that are invariant under various natural types of circular symmetry. A core motivation for this article comes from classical several complex variables where the study of maps on various types of domains is a major theme. There an important class is the *circular domains*. These behave very well under bianalytic mappings as described e.g. by Braun–Kaup–Upmeier [6].

1.1. Main results

This subsection contains a summary of the main results of the paper. Let $M(\mathbb{C})^g$ denote the sequence $(M_n(\mathbb{C})^g)_{n\in\mathbb{N}}$ of g-tuples of $n \times n$ matrices with entries from $\mathbb{C}$. A subset $\Gamma \subseteq M(\mathbb{C})^g$ is a sequence $(\Gamma(n))_n$ where $\Gamma(n) \subseteq M_n(\mathbb{C})^g$.

1.1.1. Rotationally invariant free spectrahedra

A subset $\mathcal{D} \subseteq M(\mathbb{C})^g$ is **circular** if $Z \in \mathcal{D}$ implies $e^{it} Z \in \mathcal{D}$ for all $t \in \mathbb{R}$ and is **free circular** if $UZ \in \mathcal{D}$ for each n, each $Z \in \mathcal{D}(n)$, and each $n \times n$ unitary matrix $U \in M_n(\mathbb{C})$. Here $UZ = (UZ_1, \ldots, UZ_g)$. Geometric and analytic properties of circular subsets of $\mathbb{C}^n$ and their generalizations, such as Reinhardt domains, are heavily investigated in several complex variables [22], cf. [6].

Given a tuple $A \in M_d(\mathbb{C})^g$, if there is an orthogonal decomposition of $\mathbb{C}^d$ such that with respect to this decomposition $A = A^1 \oplus A^2$, then $L_A(x) = (L_{A^1} \oplus L_{A^2})(x)$. In this case each L_{A^i} is a **subpencil** of L_A. If $\mathcal{D}_A = \mathcal{D}_{A^i}$, then L_{A^i} is a **defining subpencil** for $\mathcal{D}_A$. Say the pencil L_A is a **minimal defining pencil** for $\mathcal{D}_A$ if no proper subpencil of L_A is a defining subpencil for D_A.

Theorem 1.1 below says the tuple A in a minimal defining pencil L_A of a circular free spectrahedron is (up to unitary equivalence) block superdiagonal. It also says, if the domain is free circular, then there are just two blocks. We refer to such a domain as a **matrix pencil ball**.

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