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The behavior of fixed point free nonexpansive mappings in geodesic spaces

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A R T I C L E I N F O

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A B S T R A C T

Dropping the existence of fixed points of a nonexpansive mapping is an interesting and unusual task in metric fixed point theory. Hyperbolic geometry proved to be very relevant in the study of the behavior of fixed point free nonexpansive mappings. In this work we generalize some of the results in that direction in geodesic spaces. More precisely, we show under which additional assumptions the Picard iterative sequence of a mapping defined on a hyperbolic geodesic space tends to a point of the boundary.

1. Introduction

For a wide class of spaces, the boundedness of the closed and convex domain of a nonexpansive self-mapping suffices to expect that T has fixed points. We meet this situation for instance in uniformly convex Banach spaces, complete CAT(κ) spaces or geodesic Ptolemy spaces (see for instance [10,11,16,14]). Otherwise, the results may be completely different. In [12,16,21–23,28] the reader may find a lot of examples of spaces where nonexpansive self-mappings have fixed points even if their domain is unbounded.

The main goal of this paper is to focus on the opposite problem. Namely, how do the fixed point free nonexpansive mappings behave? The fixed point free nonexpansive mappings defined on geodesic δ-hyperbolic spaces are in close relation to the holomorphic functions defined on the interior of bounded closed convex subsets of real and complex Banach spaces; in this case the most important problem is how such functions behave on the boundary of the domain, i.e., the Denjoy–Wolff theorem (see [7,15,16,18]). In very concrete examples of spaces it is known that without additional compactness assumptions of the domain, the Picard iterative sequence for a fixed point free nonexpansive mapping may not converge to the point on the boundary. However, for the special class of firmly nonexpansive mappings, this convergence is guaranteed, for instance, in real and complex Hilbert balls with the hyperbolic metrics as it was shown in [16]. Here we will

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focus on generalizations of such results. More precisely, we will show under which additional assumptions the Picard iterative sequence converges. In our consideration we will use horoballs, i.e., sublevels $f^{-1}(-\infty, a)$ of Busemann (and not only) functions f.

The paper is organized in the following way. Section 2 includes definitions and preliminaries emphasizing the various types of boundaries for geodesic spaces and relations between them. In Section 3 we focus on the behavior of horoballs defined for functions with similar properties as Busemann ones. In Sections 4 and 5 the previous results will be applied to show under which additional assumptions one may expect that the Picard iterative sequence $(T^n x)$ is convergent to a point on the boundary and that this convergence is independent of the choice of $x \in X$, which generalizes results so far known only in very special classes of spaces (see [15,16,18]).

2. Preliminaries

Let us suppose that (X, d) is a geodesic space, i.e., for each pair of points $x, y \in X$ there is an isometric embedding $c\colon [0, d(x, y)] \to X$ such that $c(0) = x$ and $c(d(x, y)) = y$. The image of c is called a metric segment and, if this embedding is unique, it is denoted by $[x, y]$. In this case for any $\alpha \in (0, 1)$ we may define the convex combination $\alpha x + (1 - \alpha)y$ as the unique point of the metric segment $[x, y]$ such that $d(x, \alpha x + (1 - \alpha)y) = (1 - \alpha)d(x, y)$. If the isometric embedding may be extended to $c\colon [0, \infty) \to X$ then its image is called the geodesic ray. Moreover, the space is said to be uniquely geodesic if the uniqueness of embedding holds for any $x, y \in X$. In the sequel we assume that spaces are uniquely geodesic. Next we recall definitions of some subclasses of uniquely geodesic spaces.

Definition 2.1. X is said to be a Busemann space if for each triple of points $x, y, z \in X$ and metric segments $[x, y]$ and $[x, z]$ the inequality

$$d(\alpha x + (1 - \alpha)y, \alpha x + (1 - \alpha)z) \le (1 - \alpha)d(y, z) \tag{2.1}$$

is satisfied for any $\alpha \in [0, 1]$.

Remark 2.1. Sometimes the space with the property (2.1) is called Busemann convex or hyperbolic (compare with [20,26]).

As a natural example of the Busemann spaces one may consider Hilbert spaces or, more general, the so-called CAT(0) spaces. Let us recall the definition of this class of spaces. Let $\Delta(x_1, x_2, x_3)$ be the triangle consisting of a triple of points $x_1, x_2, x_3 \in X$ (its vertices) and three metric segments joining the vertices (these will be the edges of the triangle). Then one may find a comparison triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane such that $d(x_i, x_j) = \|\bar{x}_i - \bar{x}_j\|$, $i, j \in \{1, 2, 3\}$. We say that $\Delta(x_1, x_2, x_3)$ satisfies the CAT(0) inequality if for all $p, q \in \Delta(x_1, x_2, x_3)$ and their comparison points $\bar{p}, \bar{q} \in \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ the following condition

$$d(p, q) \le \|\bar{p} - \bar{q}\| \tag{2.2}$$

holds.

Definition 2.2. X is said to be a CAT(0) space if for each triangle $\Delta(x, y, z)$ with $x, y, z \in X$ the CAT(0) inequality (2.2) holds.

CAT(0) spaces satisfy also the following property which will play a crucial role in our considerations.